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Mixed Hodge structures and equivariant sheaves on the projective plane

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We describe an equivalence of categories between the category of mixed Hodge structures and a category of equivariant vector bundles on a toric model of the complex projective plane which verify some semistability condition. We then apply this correspondence to define an invariant which generalizes the notion of $\mathbf{R}$-split mixed Hodge structure and give calculations for the first group of cohomology of possibly non smooth or non-complete curves of genus 0 and 1. Finally, we describe some extension groups of mixed Hodge structures in terms of equivariant extensions of coherent sheaves.

1 Introduction

The purpose of this note is to give a geometric equivalent of the notion of mixed Hodge structure. A mixed Hodge structure is, roughly speaking, the data of a vector space endowed with three ordered filtrations which are in a certain relative position called *opposed*. There are two equivalent ways to associate to it an equivariant vector bundle on the projective plane. One can apply, on a toric model of $\mathbf{P}^2$, the general correspondence due to Klyachko [10], which gives an equivalence of categories between the category of equivariant vector bundles on a toric variety and certain sets of filtrations on vector spaces. On the other hand, one can adopt the Rees's philosophy of associating a graded ring to a ring filtered by a chain of ideals. Starting with a trifiltered vector space, we construct, for each pair of filtrations, a Rees module whose associated coherent sheaf on the standard affine open set of the projective plane is locally free and equivariant for the natural action of a 2-dimensional torus. The descriptions agree on the intersections of the affine planes, which yields an equivariant locally free sheaf on $\mathbf{P}^2$, called the *Rees bundle* associated with the trifiltered vector space. This construction, together with its inverse, yields an equivalence of categories.

This equivalence still holds when one enriches the structure of the objects involved. The geometric translation for the filtrations to be opposed is a strong semistability condition for the corresponding vector bundles. It is worth noting that no new vector bundle on the projective plane arises in this way since these bundles are instanton bundles in the sense of [13]. We do not deal with this last point here.

The idea of performing the construction of Rees bundles, which involves the three filtrations underlying a mixed Hodge structure, has a double origin: Simpson's construction of mixed twistor structures associated to the Hodge and the conjugate filtrations on the projective line in [20], [21], and Sabbah's construction of Frobenius manifolds in [18], where families of vector bundles on $\mathbf{P}^1$ associated to the Hodge and weight filtrations of a variation of mixed Hodge structure are involved.

The motivation for handling the three filtrations simultaneously is twofold. A Rees bundle on $\mathbf{P}^2$ takes into account the relative position of the Hodge filtration and its conjugate and, on the other hand, it captures all the extension data of the mixed Hodge structure contained in the weight filtration.

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Beyond his theoretical interest, the translation of the notion of mixed Hodge structure into the language of coherent sheaves aims at being the first step towards an understanding of limits of variations of Hodge structure in terms of compactification of moduli spaces of sheaves. The next step, namely a correspondence between variations of mixed Hodge structures and families of Rees bundles, will be the subject of a forthcoming paper.

This paper is organised in six parts. In Section 2, we recall equivalences between categories of equivariant coherent sheaves and categories of sets of filtrations. The core of this work is Section 3, where we translate geometrically special structures with which the filtrations are endowed. Section 4, in which we define the $\mathbf{R}$-split level, is a direct application of Section 3 to filtered vector spaces arising from Hodge theory. Some computations of $\mathbf{R}$-split levels are done in Section 5, and Section 6 deals with extensions of mixed Hodge structures.

2 Filtrations and equivariant sheaves

Let k be an algebraically closed field and let V be a finite-dimensional k-vector space endowed with n decreasing filtrations $F_1^\bullet, F_2^\bullet, \dots, F_n^\bullet$. The object $(V, F_1^\bullet, F_2^\bullet, \dots, F_n^\bullet)$ is called a *n-filtered vector space*. A filtration $F^\bullet$ of a vector space V is said to be *complete* if there exists two integers m, n such that $F^m = V$ and $F^n = \{0\}$. All the filtrations we shall consider will be complete.

A morphism $f : (V, F_1^\bullet, F_2^\bullet, \dots, F_n^\bullet) \to (V', G_1^\bullet, G_2^\bullet, \dots, G_n^\bullet)$ between two n-filtered vector spaces is a morphism of vector spaces $f : V \to V'$ which is *filtered*, or *compatible with the filtrations*, namely, for any integers i and p, $f\big(F_i^p\big) \subset G_i^p$.

We shall denote by $\mathcal{C}_{nfiltr}$ the category whose objects are finite-dimensional n-filtered vector spaces and whose morphisms are filtered morphisms. We will particularly focus on the case $n = 3$.

2.1 Equivariant sheaves on toric varieties

We denote by $\mathbf{P}_k^2$ a toric model of the projective plane and by $\mathbf{T}$ the algebraic torus which acts on it. The starting point of the correspondence we shall establish is the

Theorem 2.1 ([10], [14], [16]) *There is an equivalence of categories between the category of finite-dimensional vector spaces endowed with* 3 *complete decreasing filtrations, $\mathcal{C}_{3filtr}$, and the category whose objects are equivariant vector bundles of finite rank and whose morphism are equivariant morphisms,* $\mathrm{Bun}\big(\mathbf{P}_k^2/\mathbf{T}\big)$.

Although independently proven in [15] by using Rees's construction, which we shall detail further on, this theorem is a direct application to the projective plane of Klyachko's previous correspondence, generalized by Perling, between equivariant vector bundles on toric varieties and certain sets of filtrations of vector spaces (see [10, Theorem 2.2.1] and [16, Theorem 5.19]).

In order to establish in the next section the parallel with the Rees construction on the projective plane the author used in [15], we briefly describe this correspondence with the formalism derived in [16]. Let X_Δ be a toric variety associated to some fan Δ (we refer to [8] for notations and basic facts on toric varities), namely, here, a normal reduced and separated scheme of finite type over $\mathrm{Spec}(k)$ which contains an algebraic torus $\mathbf{T}$ as an open dense subset such that the action of the torus on itself by multiplication extends to an action of the algebraic group $\mathbf{T}$ on X_Δ. We denote by M the character group of the torus, which we identify with $\mathbf{Z}^{\dim \mathbf{T}}$. Let $\mathcal{E}$ be an equivariant quasicoherent sheaf over X_Δ. Then, on each affine $\mathbf{T}$-invariant open subset U_σ, $\sigma \in \Delta$, the action induces a decomposition of the module of section into $\mathbf{T}$-eigenspaces

$$\Gamma(U_\sigma, \mathcal{E}) = \bigoplus_{m \in M} \Gamma(U_\sigma, \mathcal{E})_m .$$

The structure of the semigroup of M associated with σ, σ_M, induces a preorder on M by setting $m \leq_{\sigma_M} m'$ iff $m' - m \in \sigma_M$. The module structure over the coordinate ring of U_σ, $k[\sigma_M]$, yields maps $\Gamma(U_\sigma, \mathcal{E})_m \to \Gamma(U_\sigma, \mathcal{E})_{m'}$ by multiplication by the character $\chi(m' - m)$ provided $m \leq_{\sigma_M} m'$. The vector spaces $\Gamma(U_\sigma, \mathcal{E})_m$ together with the morphisms given by the characters form a directed family of vector spaces with respect to the preorder which is called a *σ-family* in [16].

Each $\sigma \in \Delta$ furnishes such a family and, reciprocally, a set of σ-families associated with all $\sigma \in \Delta$ gives a system of sheaves which glue to form an equivariant sheaf on X_Δ, provided they fulfill certain compatibility conditions. Such a set of σ-families is referred to as a *Δ-family*. This construction provides an equivalence of categories between equivariant quasicoherent sheaves over X_Δ and Δ-families [16, Theorem 5.9.]

It turns out that if $\mathcal{E}$ is coherent and torsion free, the eigenspaces $\Gamma(U_\sigma, \mathcal{E})_m$ are finite-dimensional and all the morphisms involved in the corresponding Δ-family are injective, which allows us to express it in terms of (increasing) filtrations of a certain vector space. Moreover, equivariant reflexive sheaves, and hence locally free sheaves on toric curves or surfaces, correspond to vector spaces with complete filtrations associated with each *ray* in Δ, namely with each cone of dimension 1 of the fan Δ [16, Theorem 5.19].

2.2 Rees construction

Let $(V, F_1^\bullet, F_2^\bullet)$ be an object of C_{2filtr}. The $k[x,y]$-submodule of the $k[x,y]$-module $k[x^{\pm 1}, y^{\pm 1}] \otimes_k V$ generated by the elements of the form $x^{-p}y^{-q}.v$, where $v \in F_1^p \cap F_2^q$, is called the *Rees module* associated with $(V, F_1^\bullet, F_2^\bullet)$ and is denoted by $R(V, F_1^\bullet, F_2^\bullet)$. Since one can always find a splitting of a 2-filtered vector space which is compatible with both filtrations, $R(V, F_1^\bullet, F_2^\bullet)$ is free. Hence, the associated coherent sheaf by the functor $\sim$ on $\mathbf{A}_k^2 = \operatorname{Spec} k[x,y]$ is locally free. It is the subsheaf of $j_*(V \otimes \mathcal{O}_U)$ generated by the sections of the form $x^{-p}y^{-q}.v$, where $v \in F_1^p \cap F_2^q$ and $j : U = \operatorname{Spec} k[x^{\pm 1}, y^{\pm 1}] \to \operatorname{Spec} k[x,y]$ is the inclusion map.

Moreover, the comodule action of $k[s^{\pm 1}, t^{\pm 1}]$ on $R(V, F_1^\bullet, F_2^\bullet)$, given by the morphism

$$R(V, F_1^\bullet, F_2^\bullet) \longrightarrow k[s^{\pm 1}, t^{\pm 1}] \otimes_k R(V, F_1^\bullet, F_2^\bullet) \quad \text{defined by} \quad x^p y^q.v \longmapsto t^p s^q \otimes x^p y^q.v$$

allows us to endow the corresponding locally free sheaf with an action of the torus $\mathbf{G}_m^2 = \operatorname{Spec} k[s^{\pm 1}, t^{\pm 1}]$.

Note that, once endowed with the action of the affine group $\mathbf{G}_m^2$, the affine variety $\mathbf{A}_k^2$ is isomorphic to the affine toric variety associated with the (convex) cone σ in $N_\mathbf{R} = N \otimes_\mathbf{Z} \mathbf{R}$ generated by the rays $\rho_0 = (1,0)$ and $\rho_1 = (0,1)$, where $N = \mathbf{Z}^2$. Consider now the pair of increasing filtrations $(F_\bullet^1, F_\bullet^2)$ of V canonically associated with $(F_1^\bullet, F_2^\bullet)$ by letting, for each $p \in \mathbf{Z}$, $F_p^i = F_i^{-p}$, $i \in \{1,2\}$. The bifiltered vector space $(V, F_\bullet^1, F_\bullet^2)$ corresponds to the data of a family of complete filtrations associated which each ray in σ.

Lemma 2.2 *The locally free sheaves on $\mathbf{A}_k^2$ associated with, on the one hand, the σ-family, and, on the other, the Rees module which corresponds to the bifiltered vector space $(V, F_1^\bullet, F_2^\bullet)$, are isomorphic as equivariant sheaves for the action of $\mathbf{G}_m^2$.*

P r o o f. Remark that both locally free sheaves are respectively isomorphic to $\left(\bigoplus_{m_i \in C} \chi(m_i).k[\sigma_M]^{k(m_i)}\right)^\sim$ and $\left(\bigoplus_{m_i \in C} k[x,y](-m_i)^{k(m_i)}\right)^\sim$, which are isomorphic; here C is the finite set of characters m_i of the torus such that $k(m_i) = \dim_k Gr^{F_\bullet^1}_{\langle m_i, \rho_0\rangle} Gr^{F_\bullet^2}_{\langle m_i, \rho_1\rangle} V = \dim_k Gr_{F_1^\bullet}^{-\langle m_i, \rho_0\rangle} Gr_{F_2^\bullet}^{-\langle m_i, \rho_1\rangle} V \neq 0$, where $\langle\, , \,\rangle$ refers to the canonical pairing between N and its dual lattice M. The fact that the actions of the torus coincide provides the result. $\square$

Consider now the fan Δ in $N_\mathbf{R}$ generated by the rays ρ_0, ρ_1 and $\rho_2 = (-1,-1)$. The associated toric variety is the projective plane $\mathbf{P}_k^2$. Let $\mathbf{T}$ be the group acting on it and let $(V, F_0^\bullet, F_1^\bullet, F_2^\bullet)$ be an object of $\mathcal{C}_{3filtr}$. When associating with it the corresponding vector space endowed with increasing filtrations, one gets a Δ-family of complete filtrations asscociated with each ray in Δ which corresponds to an equivariant locally free sheaf on $\mathbf{P}_k^2$.

On the other hand, one can perform the above Rees construction for each pair of filtrations $(F_i^\bullet, F_j^\bullet)$, $i < j$. This yields an equivariant sheaf on each standard affine open set $\mathbf{A}_{ij}^2 = \operatorname{Spec} k\left[\frac{u_i}{u_l}, \frac{u_j}{u_l}\right]$, $i < j$, $\{i,j,l\} = \{0,1,2\}$, of the projective plane $\mathbf{P}_k^2 = \operatorname{Proj} k[u_0, u_1, u_2]$. According to Lemma 2.2, since the σ-families associated with each pair of filtrations form a Δ-family, these locally free sheaves glue together and give an equivariant locally free sheaf isomorphic to the one yielded by the Klyachko-Perling correspondence. Following [15], we shall call it the *Rees bundle* associated with $(V, F_0^\bullet, F_1^\bullet, F_2^\bullet)$ and denote it by $\xi(V, F_0^\bullet, F_1^\bullet, F_2^\bullet)$.

Remark 2.3 For each pair of integers (i,j) as above, the torus action on the Rees bundles provides a canonical isomorphism between the fiber over the intersection of the globally invariant divisors $P_{ij} = \mathbf{P}_i^1 \cap \mathbf{P}_j^1$, where $\mathbf{P}_l^1$ refers to the toric divisor associated with ρ_l, $l \in \{0,1,2\}$, whose equation is $u_l = 0$, and the bigraded vector spaces which corresponds to the filtrations $F_i^\bullet$ and $F_j^\bullet$, $\xi(V, F_0^\bullet, F_1^\bullet, F_2^\bullet)_{P_{ij}} \cong \bigoplus_{p,q} Gr^p_{F_i^\bullet} Gr^q_{F_j^\bullet} \cong \bigoplus_{p,q} Gr^p_{F_j^\bullet} Gr^q_{F_i^\bullet}$. So, the total space of the Rees bundle may be considered as a space of deformation of the vector space V into the bigraded pieces the filtrations give.

3 Geometric characterization of opposed filtrations

3.1 Opposed filtrations and semistability

Filtrations involved in Hodge theory are in a specific relative position.

Definition 3.1 ([3]) Two filtrations $F_1^\bullet, F_2^\bullet$ on a vector space V are *n-opposed* if

$$Gr^p_{F_1^\bullet} Gr^q_{F_2^\bullet} V = 0 \quad \text{unless} \quad p + q = n.$$

Three ordered filtrations $(F_0^\bullet, F_1^\bullet, F_2^\bullet)$ on V are *opposed* if

$$Gr^p_{F_1^\bullet} Gr^q_{F_2^\bullet} Gr^n_{F_0^\bullet} V = 0 \quad \text{unless} \quad p + q + n = 0.$$

Remark that three ordered filtrations $(F_0^\bullet, F_1^\bullet, F_2^\bullet)$ on V are opposed if and only if, for each integer r, $F_1^\bullet$ and $F_2^\bullet$ induce $-r$-opposed filtrations on $Gr^r_{F_0^\bullet} V$.

Remark 3.2 Recall that, by Zassenhaus's lemma (see [3, 1.2.1]), the objects $Gr^p_{F_1^\bullet} Gr^q_{F_1^\bullet} V$ and $Gr^q_{F_2^\bullet} Gr^p_{F_1^\bullet} V$ are canonically isomorphic. We emphasize the fact that a triple of opposed filtrations is *ordered*. Indeed, in $Gr^p_{F_1^\bullet} Gr^q_{F_2^\bullet} Gr^n_{F_0^\bullet} V$, $F_1^\bullet$ and $F_2^\bullet$ play a symmetrical role but neither $F_0^\bullet$ and $F_1^\bullet$ nor $F_0^\bullet$ and $F_2^\bullet$ do.

A *bigrading* of a 2-filtered vector space $(V, F_1^\bullet, F_2^\bullet)$ is a direct sum decomposition $V = \bigoplus_{p,q} V^{p,q}$ which verifies $F_1^p = \bigoplus_{(p',q'),p' \geq p} V^{p',q'}$ and $F_2^q = \bigoplus_{(p',q'),q' \geq q} V^{p',q'}$. A 2-filtered vector space always admits a bigrading.

We now introduce the notion of $\mathbf{P}^1_0$-semistability which is the geometric equivalent for the Rees bundles to the property to be opposed for the corresponding triples of filtrations. Let $\mathcal{E}$ be a coherent sheaf on a smooth projective variety. The slope of $\mathcal{E}$ is the ratio

$$\mu(\mathcal{E}) = \deg(\mathcal{E})/\mathrm{rk}(\mathcal{E})$$

if $\mathrm{rk}(\mathcal{E}) > 0$, and is defined to be $\mu(\mathcal{E}) = 0$ otherwise. A coherent sheaf $\mathcal{E}$ is *μ-semistable* if for every coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ we have

$$\mu(\mathcal{F}) \leq \mu(\mathcal{E}).$$

Let $j : \mathbf{P}^1_0 \hookrightarrow \mathbf{P}^2_k$ be the inclusion morphism.

Definition 3.3 A locally free sheaf $\mathcal{E}$ on $\mathbf{P}^2_k$ is *$\mathbf{P}^1_0$-semistable* if $j^*\mathcal{E}$, its restriction to the line $\mathbf{P}^1_0$, is μ-semistable as a locally free sheaf on the projective line.

Like locally free sheaves on the projective line split into a sum of line bundles, a locally free sheaf on $\mathbf{P}^2_k$ is $\mathbf{P}^1_0$-semistable if and only if its restriction to $\mathbf{P}^1_0$ is the direct sum of line bundles of the same slope.

Since j^* induces a monomorphism from $H^2(\mathbf{P}^2_k, \mathbf{Z})$ to $H^2(\mathbf{P}^1_0, \mathbf{Z})$ and the degree is functorial, the $\mathbf{P}^1_0$-semistability is a stronger notion than the μ-semistability. Let $\mathcal{E}$ be a coherent sheaf on $\mathbf{P}^2_k$, we thus have

$$\mathcal{E} \text{ is } \mathbf{P}^1_0\text{-semistable} \Longrightarrow \mathcal{E} \text{ is } \mu\text{-semistable}.$$

Let $\omega \in H^2(\mathbf{P}^2_k, \mathbf{Z})$ be the cohomology class of a hyperplane.

Proposition 3.4 *Let $\xi(V, F_0^\bullet, F_1^\bullet, F_2^\bullet)$ be the Rees vector bundles on $\mathbf{P}^2_k$ associated with a trifiltered vector space whose filtrations are opposed, $(V, F_0^\bullet, F_1^\bullet, F_2^\bullet) \in \mathcal{C}_{3filtr,opp}$. Then,*

(i) *$\xi(V, F_0^\bullet, F_1^\bullet, F_2^\bullet)$ is $\mathbf{P}^1_0$-semistable,*

(ii) *$\mathrm{c}_1(\xi(V, F_0^\bullet, F_1^\bullet, F_2^\bullet)) = 0$, and,*

(iii) *$\mathrm{c}_2(\xi(V, F_0^\bullet, F_1^\bullet, F_2^\bullet)) = \frac{1}{2}\sum_{p,q}(h^{p,q} - s^{p,q})(p+q)^2\omega^2$,*
where, for each pair of integers (p, q), $h^{p,q} = \dim_k Gr^q_{F_2^\bullet} Gr^p_{F_1^\bullet} Gr^{-p-q}_{F_0^\bullet} V$ and $s^{p,q} = \dim_k Gr^q_{F_2^\bullet} Gr^p_{F_1^\bullet} V$.

The calculation of the second Chern class in the above proposition will be useful in the next section. We shall denote by $T(V)^\bullet$, or by $T^\bullet$ when the context is clear, the trivial filtration of V given by $T(V)^0 = V$ and $T(V)^1 = \{0\}$, and by $F[k]^\bullet$ the k-shifted filtration associated with $F^\bullet$ defined by, for each integer p, $F[k]^p = F^{k+p}$.

Proof. When one restricts a Rees bundle to a divisor $\mathbf{P}^1_i$, $i \in \{0,1,2\}$, one obtains the graded pieces associated with the filtration $F_i^\bullet$. So, the restrictions of the Rees bundles $\xi\big(\bigoplus_r \big(Gr^r_{F_0^\bullet}V, T\big(Gr^r_{F_0^\bullet}V\big)[-r]^\bullet, F_1^\bullet, F_2^\bullet\big)\big)$ and $\xi\big(V, F_0^\bullet, F_1^\bullet, F_2^\bullet\big)$ to $\mathbf{P}^1_0$ are isomorphic (here we have used the same notations for the filtrations induced by $F_1^\bullet$ and $F_2^\bullet$ on the graded pieces). The second one is trivial since we have a grading compatible with all the filtrations, hence $\xi(V, F_0^\bullet, F_1^\bullet, F_2^\bullet)|_{\mathbf{P}^1_0} \cong \mathcal{O}^{\dim_k V}_{\mathbf{P}^1_0}$, which proves (i) and (ii).

To prove (iii), we proceed in several steps in order to reduce the computation of the Chern classes to those of lines bundles. Let $\pi : \widetilde{\mathbf{P}}^2_k \to \mathbf{P}^2_k$ be the blowing-up of the projective plane at P_{12} and let E be the exceptional curve. $\widetilde{\mathbf{P}}^2_k$ is the toric variety associated to the fan Δ_E obtained by adding to Δ a ray ρ_E in σ_0 generated by $n(\rho_E) = (1,1)$ in N. We denote by σ_0' and σ_0'' the 2-dimensional cones obtained from σ_0. The toric divisor associated to ρ_E is the projective line E. Consider now the Δ_E-family of complete filtrations associated to each ray in Δ_E, $(V, F_0^\bullet, F_1^\bullet, T(V)^\bullet, F_2^\bullet)$. Let $\xi_E(V, F_0^\bullet, F_1^\bullet, T(V)^\bullet, F_2^\bullet)$ be the corresponding locally free sheaf on the toric surface $\widetilde{\mathbf{P}}^2_k$.

Here we compare $\pi^*\xi(V, F_0^\bullet, F_1^\bullet, F_2^\bullet)$ to $\xi_E(V, F_0^\bullet, F_1^\bullet, T(V)^\bullet, F_2^\bullet)$. By construction, both coincide on $\widetilde{\mathbf{P}}^2_k \setminus E$. Since it is equivariant, locally free and of finite rank, according to the above-mentioned correspondence, $\pi^*\xi(V, F_0^\bullet, F_1^\bullet, F_2^\bullet)$ corresponds to a Δ_E-family, which is of the form $(V, F_0^\bullet, F_1^\bullet, G^\bullet, F_2^\bullet)$ because of its description in the complementary of E. When one explicits the Rees sheaves on $U_{\sigma_0'}$ and $U_{\sigma_0''}$ corresponding to the pullback of the restriction of $\xi(V, F_0^\bullet, F_1^\bullet, F_2^\bullet)$ to the chart $\mathbf{A}^2_0$ in $\mathbf{P}^2_k$, one gets $G^\bullet = F_1^\bullet \star F_2^\bullet$, where the convolution is defined by

$$G^r = \sum_{p+q \geq r} F_1^p \cap F_2^q.$$

Suppose now that $F_1^\bullet$ and $F_2^\bullet$ are *positive*, namely $F_1^0 = F_2^0 = V$; this can always be realised, without changing the (no equivariant) isomorphism class of the bundle, by shifting the indices of $F_1^\bullet$ and $F_2^\bullet$ to obtain $F_1^0 = F_2^0 = V$ and, next, by shifting the indices of $F_0^\bullet$ in order to keep the filtrations opposed. $G^\bullet$ is now positive and, therefore, there is a filtered morphism from $(V, T(V)^\bullet)$ to $(V, G^\bullet)$. This morphism induces an injective morphism of equivariant locally free sheaves whose cokernel's support is included in E

$$0 \longrightarrow \xi_E(V, F_0^\bullet, F_1^\bullet, T(V)^\bullet, F_2^\bullet) \longrightarrow \pi^*\xi(V, F_0^\bullet, F_1^\bullet, F_2^\bullet) \longrightarrow \mathcal{T}_E \longrightarrow 0. \tag{3.1}$$

Since, contrary to the starting Rees bundle on $\mathbf{P}^2_k$, $F_1^\bullet$ and $F_2^\bullet$ are never involved together on the same globally invariant affine open set in the construction of $\xi_E(V, F_0^\bullet, F_1^\bullet, T(V)^\bullet, F_2^\bullet)$, one can cut it off by using $F_0^\bullet$. Hence, if $V' = F_0^p$ for some p and $V'' = V/V'$ is the quotient, on $\widetilde{\mathbf{P}}^2_k$, we have, without changing the notation for the induced filtrations,

$$0 \longrightarrow \xi_E(V', F_0^\bullet, F_1^\bullet, T(V')^\bullet, F_2^\bullet) \longrightarrow \xi_E(V, F_0^\bullet, F_1^\bullet, T(V)^\bullet, F_2^\bullet) \longrightarrow \xi_E(V'', F_0^\bullet, F_1^\bullet, T(V'')^\bullet, F_2^\bullet) \longrightarrow 0.$$

One can repeat the process until we get vector bundles of the form $\xi_E\big(Gr^r_{F_0^\bullet}V, F_0^\bullet, F_1^\bullet, T^\bullet, F_2^\bullet\big)$ in which only two filtrations are involved, and, hence, which splits into a direct sum of line bundles. This gives

$$\begin{aligned}
\mathrm{ch}(\xi_E(V, F_0^\bullet, F_1^\bullet, T(V)^\bullet, F_2^\bullet)) &= \sum_r \mathrm{ch}\big(\xi_E\big(Gr^r_{F_0^\bullet}V, T\big(Gr^r_{F_0^\bullet}V\big)[-r]^\bullet, F_1^\bullet, T\big(Gr^r_{F_0^\bullet}V\big)^\bullet, F_2^\bullet\big)\big)\\
&= \sum_{r,p,q} \dim_k\big(Gr^q_{F_2^\bullet}Gr^p_{F_1^\bullet}Gr^r_{F_0^\bullet}V\big)\,\mathrm{ch}\big(\xi_E\big(k, T[-r]^\bullet, T[-p]^\bullet, T^\bullet, T[-q]^\bullet\big)\big)\\
&= \dim_k V - \frac{1}{2}\sum_{p,q} h^{p,q}(p+q)^2\widetilde{\omega}^2
\end{aligned}$$

since the filtrations are opposed, and, because of the formula $\mathrm{c}_1(\xi_E(k, T[-r]^\bullet, T[-p]^\bullet, T^\bullet, T[-q]^\bullet)) = (r+p+q)\widetilde{\omega}$, we have $\mathrm{ch}(\xi_E(k, T[-r]^\bullet, T[-p]^\bullet, T^\bullet, T[-q]^\bullet) = 1 + (r+p+q)\widetilde{\omega} + \frac{1}{2}(r^2 + 2rp + 2rq)\widetilde{\omega}^2$ where $\widetilde{\omega}$ is the pullback of ω in $H^2(\widetilde{\mathbf{P}}^2_k, \mathbf{Z})$.

Let us now compute the Chern character of $\mathcal{T}_E$. This coherent sheaf is supported on E and do not depend on $F_0^\bullet$. Thus, we can rewrite the exact sequence (3.1) using $F_1^\bullet$ $\big($or $F_2^\bullet\big)$ instead of $F_0^\bullet$

$$0 \longrightarrow \xi_E(V, F_1^\bullet, F_1^\bullet, T^\bullet, F_2^\bullet) \longrightarrow \pi^*\xi(V, F_1^\bullet, F_1^\bullet, F_2^\bullet) \longrightarrow \mathcal{T}_E \longrightarrow 0.$$

Both vector bundles $\xi_E(V, F_1^\bullet, F_1^\bullet, T^\bullet, F_2^\bullet)$ and $\pi^*\xi(V, F_1^\bullet, F_1^\bullet, F_2^\bullet)$ split into a sum of line bundles. The Chern character of the first is given by the above formula

$$\mathrm{ch}(\xi_E(V, F_1^\bullet, F_1^\bullet, T^\bullet, F_2^\bullet)) \;=\; \dim_k V + \sum_{p,q} s^{p,q}\left((2p+q)\widetilde{\omega} + \frac{1}{2}(3p^2+2pq)\widetilde{\omega}^2\right).$$

For the second,

$$\begin{aligned}\mathrm{ch}(\pi^*\xi(V, F_1^\bullet, F_1^\bullet, F_2^\bullet)) \;&=\; \pi^*\mathrm{ch}\left(\bigoplus_{p,q}\xi\big(V, T[p]^\bullet, T[p]^\bullet, T[q]^\bullet\big)^{s^{p,q}}\right)\\ &=\; \dim_k V + \sum_{p,q} s^{p,q}\left((2p+q)\widetilde{\omega} + \frac{1}{2}(4p^2+q^2+4pq)\widetilde{\omega}^2\right).\end{aligned}$$

This gives $\mathrm{ch}(\mathcal{T}_E) \;=\; \frac{1}{2}\sum_{p,q} s^{p,q}(p+q)^2\widetilde{\omega}^2$, which allows us to conclude since the Chern character is additive. □

Remark 3.5 Since the condition for three filtrations to be opposed is not symmetrical, they do not play the same role in the formula giving the Chern classes.

Theorem 3.6 *The Rees construction establishes an equivalence of categories between the category of finite dimensional* 3*-filtered* k*-vector spaces whose ordered filtrations are opposed and the category of* $\mathbf{T}$*-equivariant* $\mathbf{P}^1_0$*-semistable vector bundles of degree* 0 *on the projective plane:*

$$\mathcal{C}_{3filtr,opp} \rightleftarrows \mathrm{Bun}_{\mathbf{P}^1_0\text{-}semistable,\mu=0}\big(\mathbf{P}^2_k/\mathbf{T}\big)\,.$$

P r o o f. According to the preceding proposition a Rees bundles associated to a vector space endowed with opposed filtrations is $\mathbf{P}^1_0$-semistable and has zero first Chern class.

Reciprocally, suppose we are given a $\mathbf{T}$-equivariant $\mathbf{P}^1_0$-semistable degree 0 vector bundle $\mathcal{E}$ on $\mathbf{P}^2_k$. Let $(V, F_0^\bullet, F_1^\bullet, F_2^\bullet)$ be the associated element in $\mathcal{C}_{3filtr}$. Suppose there exists a triple $(r_0, p_0, q_0) \in \mathbf{Z}^3$ such that $r_0 + p_0 + q_0 > 0$ and $Gr^{p_0}_{F_1^\bullet} Gr^{q_0}_{F_2^\bullet} Gr^{r_0}_{F_0^\bullet} V \neq \{0\}$. Then, there exists a one-dimensional subvector space $V' \subset V$ whose projection on $Gr^{p_0}_{F_1^\bullet} Gr^{q_0}_{F_2^\bullet} Gr^{r_0}_{F_0^\bullet} V$ is not zero. Let $(V', T^\bullet[r_0], T^\bullet[p_0], T^\bullet[q_0])$ be the trifiltered vector space whose filtrations are defined by $T^p[i] = V'$ if $p \leq i$ and $T^p[i] = 0$ otherwise. The monomorphism of Rees $k[u_0, u_1, u_2]$-modules induces an injective map of locally free sheaves

$$0 \longrightarrow \xi(V', T^\bullet(r_0), T^\bullet(p_0), T^\bullet(q_0)) \longrightarrow \mathcal{E}\,.$$

By the formula given in the proof of Proposition 3.4

$$c_1(\xi(V', T^\bullet(r_0), T^\bullet(p_0), T^\bullet(q_0))) = r_0 + p_0 + q_0 > 0,$$

which contradicts the μ-semistability and hence the $\mathbf{P}^1_0$-semistability of $\mathcal{E}$. Moreover,

$$\mathrm{c}_1(\mathcal{E}) = \sum_{r,p,q} \dim_k Gr^q_{F_2^\bullet} Gr^p_{F_1^\bullet} Gr^r_{F_0^\bullet} V(r+p+q)\omega.$$

The preceding fact proves that there is no positive contribution to the first Chern class of $\mathcal{E}$ in this formula.

Every triple (r_0, p_0, q_0) such that $r_0 + p_0 + q_0 < 0$ and $Gr^{p_0}_{F_1^\bullet} Gr^{q_0}_{F_2^\bullet} Gr^{r_0}_{F_0^\bullet} V \neq 0$ will now give a negative contribution to the first Chern class which is zero; therefore, such a space does not exist, which proves the theorem. □

3.2 Categories of semistable reflexive sheaves

We recall some facts from [14]. Let $\mathcal{R}efl_\mu(X)$ be the category whose objects are equivariant μ-semistable reflexive sheaves on a nonsingular algebraic variety X (we refer to [9] for the basic properties about reflexive sheaves). Let $\mathcal{F}$ be a coherent sheaf on X and consider the canonical morphism to its double dual $\nu : \mathcal{F} \to \mathcal{F}^{**}$.

The sheaf $\mathcal{F}^{**}$ is reflexive. It is called the *reflexive sheaf associated* to $\mathcal{F}$ and is canonically isomorphic to it when $\mathcal{F}$ is reflexive. The kernel, cokernel, image and coimage of a morphism in the category of reflexive sheaves is defined to be the reflexive sheaf respectively associated to the kernel, cokernel, image and coimage of the morphism in the category of coherent sheaves. Let $f : \mathcal{E} \to \mathcal{F}$ be a morphism in $\mathcal{R}efl_\mu(X)$. In fact, the kernel of f when considered as a morphism of coherent sheaves is already reflexive. Associated to f we have exact sequences

$$\begin{tikzcd} 0 \arrow[r] & Ker(f) \cong Ker(f)^{**} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{F} \arrow[r] \arrow[dr] & Coker(f) \arrow[r] \arrow[d, "\nu"] & 0 \\ & & & & Coker(f)^{**} \arrow[r] & 0, \end{tikzcd} \tag{3.2}$$

where the horizontal sequence is exact in the category of coherent sheaves and the other is exact in $\mathcal{R}efl_\mu(X)$.

Suppose now X is endowed with the action of an algebraic group G and denote by $\mathcal{R}efl_\mu(X/G)$ the subcategory of $\mathcal{R}efl_\mu(X)$ whose objects and morphisms are G-equivariant.

Theorem 3.7 ([14]) *$\mathcal{R}efl_\mu(X/G)$ is an abelian category.*

This category is clearly additive. In order to prove that it is exact, in [14], we use the fact that μ-semistability allows us to exhibit an isomorphism between the image and the coimage of a morphism in the complement of a subvariety of codimension at least 2. Since reflexive sheaves are normal, this provides an isomorphism between them.

Remark 3.8 It should be noted that since cokernels in both categories might not agree, $\mathcal{R}efl_\mu(X/G)$ is not a sub-abelian category of the category of equivariant coherent sheaves on X in general.

Since reflexive sheaves on curves and surfaces are locally free, when considering the action of the trivial group, we recover the classical result (see [17] for example)

Corollary 3.9 *The category of μ-semistable vector bundles on a nonsingular curve or surface is abelian.*

In [14], Theorem 3.1, it was shown that even if one imposes a stronger condition of semistability than μ-semistability, namely the $\mathbf{P}^1_0$-semistability, the category of degree 0 semistable equivariant sheaves on $\mathbf{P}^2_k$ is abelian.

Proposition 3.10 ([14]) *The category of $\mathbf{T}$-equivariant $\mathbf{P}^1_0$-semistable vector bundles of degree 0 on the projective plane $\mathrm{Bun}_{\mathbf{P}^1_0\text{-semistable},\mu=0}(\mathbf{P}^2_k/\mathbf{T})$ is abelian.*

Thus, by Theorem 3.6, we recover, in a geometric way, Deligne's result in [3, Theorem 1.2.10].

Corollary 3.11 *The category $\mathcal{C}_{3filtr,opp}$ of finite dimensional trifiltered k-vector spaces whose ordered filtrations are opposed is abelian.*

3.3 Real structures

From now onwards, we shall work on $\mathbf{C}$. Consider the antiholomorphic involution of $\mathbf{P}^2_{\mathbf{C}}$, $\tau : (u_0, u_1, u_2) \mapsto (\overline{u}_0, \overline{u}_2, \overline{u}_1)$. The divisor $\mathbf{P}^1_0$ is globally invariant by τ.

Let $\mathcal{E}$ be an $\mathcal{O}_{\mathbf{P}^2_{\mathbf{C}}}$-module. We define the sheaf $\tau^*(\mathcal{E})$ by letting, for each Zariski open set U,

$$\tau^*\mathcal{E}(U) = \mathcal{E}(\tau(U)).$$

It is canonically endowed with an $\mathcal{O}_{\mathbf{P}^2_{\mathbf{C}}}$-module structure by setting, for each $e \in \tau^*\mathcal{E}(U)$ and each $f \in \mathcal{O}_{\mathbf{P}^2_{\mathbf{C}}}(U)$,

$$f.e = \overline{\tau^*(f)}e.$$

A *τ-equivariant* coherent sheaf on the complex projective plane is the data of a coherent sheaf $\mathcal{E}$ and of a morphism of $\mathcal{O}_{\mathbf{P}^2_{\mathbf{C}}}$-modules $f : \mathcal{E} \to \tau^*\mathcal{E}$ such that $\tau^*(f) \circ f = id_{\mathcal{E}}$.

A *$\mathbf{T}^\tau$-equivariant* coherent sheaf is a coherent sheaf that is both $\mathbf{T}$ and τ-equivariant. $\mathbf{T}^\tau$-equivariant sheaves are naturally associated with finite dimensional trifiltered complex vector spaces $(V_{\mathbf{C}}, F_0^\bullet, F_1^\bullet, F_2^\bullet)$ with underlying real structure, namely such that $V_{\mathbf{C}}$ is of the form $V_{\mathbf{C}} = V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ for some real vector space $V_{\mathbf{R}}$, and whose second and third filtrations are conjugated to one another, which means that for each integer p, $\overline{F_2^p} = F_1^p$ (here we take the conjugate with respect to the underlying real structure on $V_{\mathbf{R}}$). We denote by $\mathcal{C}_{3filtr,opp,\mathbf{R}}$ the category whose objects are of this form and whose morphisms are morphisms of real vector spaces which induce filtered morphisms when passing to the complex structure. There is a forgetful functor from $\mathcal{C}_{3filtr,opp,\mathbf{R}}$ to $\mathcal{C}_{3filtr,opp}$ which consists in forgetting the real structure.

Theorem 3.12 *The Rees construction establishes an equivalence of categories*:

$$\mathcal{C}_{3filtr,opp,\mathbf{R}} \rightleftarrows \mathrm{Bun}_{\mathbf{P}^1_0\text{-}semistable,\mu=0}\left(\mathbf{P}^2_{\mathbf{C}}/\mathbf{T}^{\tau}\right).$$

Moreover, the category $C_{3filtr,opp,\mathbf{R}}$ is abelian.

P r o o f. One immediately verifies that there is a one-to-one correspondence between τ-equivariant objects and triples of filtrations whose filtrations are conjugate. Since the functor τ^* is exact, τ being an homeomorphism, the second statement is a direct consequence of the fact that $\mathrm{Bun}_{\mathbf{P}^1_0\text{-semistable},\mu=0}\left(\mathbf{P}^2_{\mathbf{C}}/\mathbf{T}\right)$ is abelian. □

3.4 Framing

When one wants to compare filtrations on a fixed vector space, situation which arises, for example, when one deals with variations of mixed Hodge structure, one needs a stronger notion of equivalence between filtered vector spaces and sheaves than that of equivalence of categories.

Fix a finite-dimensional vector space V_0 and consider the category of triplets of complete filtrations of V_0, $\mathcal{C}_{3filtr,opp,V_0}$. Its objects are of the form $((V, F_0^\bullet, F_1^\bullet, F_2^\bullet), \varphi)$, where $(V, F_0^\bullet, F_1^\bullet, F_2^\bullet) \in \mathcal{C}_{3filtr,opp}$ and $\varphi : V \xrightarrow{\sim} V_0$ is an isomorphism, and its morphisms are morphisms in $\mathcal{C}_{3filtr,opp}$ which commute with the isomorphisms to V_0.

Consider now the category of *framed Rees bundles*, $\mathrm{Bun}_{\mathbf{P}^1_0\text{-semistable},\mu=0,V_0}\left(\mathbf{P}^2_{\mathbf{C}}/\mathbf{T}\right)$, whose objects are pairs $(\mathcal{E}, \psi)$, where $\mathcal{E} \in \mathrm{Bun}_{\mathbf{P}^1_0\text{-semistable},\mu=0}\left(\mathbf{P}^2_{\mathbf{C}}/\mathbf{T}\right)$ and $\psi : \mathcal{E}_{(1:1:1)} \xrightarrow{\sim} V_0$ is a *framing*, and whose morphisms are the morphisms in the category of sheaves which commute with the framings. We immediately verify that:

Corollary 3.13 *For each finite-dimensional vector space V_0, there is a natural one-to-one correspondence between the isomorphism classes of objects in $\mathcal{C}_{3filtr,opp,V_0}$ and in $\mathrm{Bun}_{\mathbf{P}^1_0\text{-}semistable,\mu=0,V_0}\left(\mathbf{P}^2_{\mathbf{C}}/\mathbf{T}\right)$.*

4 Mixed Hodge structures and equivariant sheaves

In this section, we apply the correspondence stated in the preceding section to filtered vector spaces arising from Hodge theory. We first recall some definitions and results about mixed Hodge structures.

Definition 4.1 A *pure $\mathbf{R}$-Hodge structure of weight r* is a triple $(H_{\mathbf{R}}, F^\bullet, \overline{F}^\bullet)$ consisting of a finite dimensional $\mathbf{R}$-vector space $H_{\mathbf{R}}$ and two decreasing filtrations of $H_{\mathbf{C}} = H_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$, the *Hodge filtration* $F^\bullet$, and its conjugate filtration with respect to the underlying real structure $\overline{F}^\bullet$, such that $F^\bullet$ and $\overline{F}^\bullet$ are r-opposed.

Definition 4.2 A *$\mathbf{R}$-mixed Hodge structure* is a quadruple $(H_{\mathbf{R}}, W_\bullet, F^\bullet, \overline{F}^\bullet)$ which consists of a finite dimensional $\mathbf{R}$-vector space $H_{\mathbf{R}}$, an increasing filtration of this real vector space $W_\bullet$ called the *weight filtration*, and two filtrations $F^\bullet$ and $\overline{F}^\bullet$ of $H_{\mathbf{C}} = H_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$, conjugate to each other with respect to the underlying real structure, such that $F^\bullet$ and $\overline{F}^\bullet$ induce a pure Hodge structure of weight r on each quotient $Gr_r^W = W_r/W_{r-1}$ or, equivalently, such that the three ordered filtrations $(W_\bullet, F^\bullet, \overline{F}^\bullet)$ of $H_{\mathbf{C}}$ are opposed (here the weight filtration is viewed as a filtration of $H_{\mathbf{C}}$).

We shall denote by $\mathbf{R}$-MHS the category whose objects are $\mathbf{R}$-mixed Hodge structures and morphisms are morphisms between real vector spaces whose associated morphisms between complex vector spaces are compatible with the filtrations.

In the same way, we can define the category of complex mixed Hodge structures, denoted by $\mathbf{C}$-MHS, by only requiring the objects to consist of a finite dimensional complex vector space endowed with three ordered opposed filtrations, the first being increasing and the other decreasing. An element $H \in \mathbf{C}$-MHS corresponds to a quadruple $(H_{\mathbf{C}}, W_\bullet, F^\bullet, \hat{F}^\bullet)$. Morphisms in $\mathbf{C}$-MHS are morphisms of complex vector spaces compatible with the filtrations.

When a definition or a result concerns both categories, we do not specify any of them.

Definition 4.3 The *length* of a mixed Hodge structure is the length of the largest interval $[a,b]$ such that $Gr_r^W \neq 0$ for each $r \in \{a,b\}$. In particular, mixed Hodge structures of length 0 are pure Hodge structures.

The *level* of a mixed Hodge structure is the length of the largest interval $[a,b]$ such that $Gr_p^F \neq 0$ for each $r \in \{a,b\}$.

A *bigrading* of a mixed Hodge structure H is a direct sum decomposition $H_{\mathbf{C}} = \bigoplus_{p,q} H^{p,q}$ of the underlying complex vector space which verifies $W_r = \bigoplus_{k+l\leq r} H^{k,l}$ and $F^p = \bigoplus_{k\geq p,l} H^{k,l}$.

Following Deligne, one obtains an analogue of the Hodge decomposition for mixed Hodge structures:

Lemma 4.4 ([3]) *Let $H \in \mathbf{R}$-MHS be a mixed Hodge structure. Then, there exists a unique bigrading of H, denoted by $\{I^{p,q}\}_{p,q}$, such that*

$$I^{p,q} \equiv \overline{I}^{q,p} \bmod W_{p+q-2}.$$

The following notion is important when one considers degenerations of mixed Hodge structures (see [11]).

Definition 4.5 A $\mathbf{R}$-mixed Hodge structure H is said to be $\mathbf{R}$-*split* if for each p,q the $I^{p,q}$ vector spaces defined in Lemma 4.4 verify $I^{q,p} = \overline{I}^{p,q}$.

A $\mathbf{C}$-mixed Hodge structure H is said to be *split* if there is a bigrading $H_{\mathbf{C}} = \bigoplus_{p,q} H^{p,q}$ compatible with the third filtration, namely such that $\hat{F}^q = \bigoplus_{k,l\geq q} H^{k,l}$.

When a $\mathbf{R}$-mixed Hodge structure H is $\mathbf{R}$-split, the $I^{p,q}$ spaces furnish a bigrading of $H_{\mathbf{C}}$ which is compatible with the third filtration, so a $\mathbf{R}$-split mixed Hodge structure is split when considered as a $\mathbf{C}$-mixed Hodge structure.

By Lemma 4.4, every mixed Hodge structure whose length is lower than 2 is $\mathbf{R}$-split. In particular, every pure Hodge structure is $\mathbf{R}$-split.

4.1 Vector bundles associated with mixed Hodge structures

We associate to each mixed Hodge structure H in $\mathbf{R}$-MHS (resp. $\mathbf{C}$-MHS) an object in $\mathcal{C}_{3filtr,opp,\mathbf{R}}$ (resp. $\mathcal{C}_{3filtr,opp}$). For this purpose, in the trifiltered vector space a mixed Hodge structures provides, we substitute the weight filtration $W_\bullet$ by its associated decreasing filtration $W^\bullet$.

Then, to each object in $\mathcal{C}_{3filtr,opp,\mathbf{R}}$ (resp. $\mathcal{C}_{3filtr,opp}$) associated with a mixed Hodge structure H corresponds a Rees bundle on the toric complex projective plane, denoted by $\xi_{\mathbf{P}^2}(H)$, as described in the preceding section.

Definition 4.6 The vector bundle $\xi_{\mathbf{P}^2}(H)$ is called the *Rees vector bundle* associated to the mixed Hodge structure H.

As a direct outcome of Theorems 3.6 and 3.12 we get

Theorem 4.7 *The category of complex mixed Hodge structures* $\mathbf{C}$-MHS *is equivalent to the category of* $\mathbf{T}$*-equivariant* $\mathbf{P}^1_0$*-semistable vector bundles of degree* 0 *on the projective plane*

$$\mathbf{C}\text{-MHS} \rightleftarrows \mathrm{Bun}_{\mathbf{P}^1_0\text{-}semistable,\mu=0}(\mathbf{P}^2/\mathbf{T}).$$

The category of real mixed Hodge structures $\mathbf{R}$-MHS *is equivalent to the category of* $\mathbf{T}^\tau$*-equivariant* $\mathbf{P}^1_0$*-semistable vector bundles of degree* 0 *on the projective plane*

$$\mathbf{R}\text{-MHS} \rightleftarrows \mathrm{Bun}_{\mathbf{P}^1_0\text{-}semistable,\mu=0}(\mathbf{P}^2/\mathbf{T}^\tau).$$

As a consequence, we recover the following fact [3, Theorem 1.2.10] in a geometric way:

Corollary 4.8 ([3]) *The category of real and complex mixed Hodge structures are abelian.*

4.2 Short exact sequences

Let $0 \longrightarrow A \xrightarrow{i} H \xrightarrow{\pi} B \longrightarrow 0$ be an exact sequence in the category of mixed Hodge structures. Here we study the possible failure of exactness of the corresponding sequence of vector bundles when considered as a sequence in the category of coherent sheaves.

The associated exact sequence in $\mathrm{Bun}_{\mathbf{P}^1_0\text{-semistable},\mu=0}(\mathbf{P}^2/\mathbf{T}^\tau)$ is

$$0 \longrightarrow \xi_{\mathbf{P}^2}(A) \xrightarrow{\ i\ } \xi_{\mathbf{P}^2}(H) \xrightarrow{\ \pi\ } \xi_{\mathbf{P}^2}(B) \longrightarrow 0.$$

By construction, $\xi_{\mathbf{P}^2}(B)$ is the reflexive sheaf associated with the cokernel of i in the category of coherent sheaves, $Coker(i)$, whose singularity set is included in P_{12}. Indeed, the singularity set is at least 2-codimensional, so included in the set of fixed points of the action, and the restriction of $Coker(i)$ to $\mathbf{P}^1_0$ is locally free. The support of the cokernel $\mathcal{T}$ of the canonical morphism $Coker(i) \to Coker(i)^{**} = \xi_{\mathbf{P}^2}(B)$, which is injective, is therefore included in P_{12}. In the category of coherent sheaves we thus have the exact sequence

$$0 \longrightarrow \xi_{\mathbf{P}^2}(A) \xrightarrow{\ i\ } \xi_{\mathbf{P}^2}(H) \xrightarrow{\ \tilde{\pi}\ } \xi_{\mathbf{P}^2}(B) \longrightarrow \mathcal{T} \longrightarrow 0. \tag{4.1}$$

4.3 New Hodge numbers and $\mathbf{R}$-split level

The fact that the Rees bundle associated with a cokernel in the category of mixed Hodge structures is not a cokernel in the category of coherent sheaves is showing beneath the behaviour of some integers denoted by $s^{p,q}$ which are similar to the Hodge numbers $h^{p,q}$; recall that the Hodge numbers are defined by, for a mixed Hodge structure H,

$$h_H^{p,q} = \dim_{\mathbf{C}} Gr_F^p Gr_{\overline{F}}^q Gr_W^{-p-q} H_{\mathbf{C}} = \dim_{\mathbf{C}} Gr_F^p Gr_W^{-p-q} H_{\mathbf{C}} = \dim_{\mathbf{C}} Gr_{\overline{F}}^q Gr_W^{-p-q} H_{\mathbf{C}}.$$

The integers $s^{p,q}$ measure in some sense the relative position of the filtrations but are not additive contrary to the Hodge numbers.

Definition 4.9 Let H be a mixed Hodge structure. We let

$$s_H^{p,q} = \dim_{\mathbf{C}} Gr_F^p Gr_{\overline{F}}^q H_{\mathbf{C}}.$$

We make the identification $H^4(\mathbf{P}^2, \mathbf{Z}) = \mathbf{Z}$.

Definition 4.10 Let H be a mixed Hodge structure. We define the $\mathbf{R}$-split level of H to be the integer

$$\alpha(H) = \mathrm{c}_2(\xi_{\mathbf{P}^2}(H)).$$

Remark 4.11 The definition makes sense for $\mathbf{C}$-MHS too. The term *split level* would be, however, more appropriate for objects in this category.

A *Tate Hodge structure of weight* k, denoted by $T\langle k\rangle$, is the unique Hodge structure of rank 1 and of pure type (k, k). Since Rees bundles associated with mixed Hodge structures are of degree 0, one easily verifies that for each mixed Hodge structures H, H' and each $k \in \mathbf{Z}$:

(1) $\alpha(H \oplus H') = \alpha(H) + \alpha(H')$,

(2) $\alpha(H^*) = \alpha(H)$, where $H^* = \mathrm{Hom}(H, T\langle 0\rangle)$ in the category of mixed Hodge structures,

(3) $\alpha(H \otimes H') = \dim_{\mathbf{C}} H'_{\mathbf{C}}\, \alpha(H) + \dim_{\mathbf{C}} H_{\mathbf{C}}\, \alpha(H')$,

(4) $\alpha(H \otimes T\langle k\rangle) = \alpha(H)$.

The following formula gives an explicit formula for the $\mathbf{R}$-split level. It is a direct consequence of Proposition 3.4.

Proposition 4.12 *The* $\mathbf{R}$*-split level is expressed by*

$$\alpha(H) = \frac{1}{2} \sum_{p,q} (p+q)^2 \left(h_H^{p,q} - s_H^{p,q}\right).$$

The categories of mixed Hodge structures involved here are abelian, hence we can define extension groups, which, endowed with Baer summation, are abelian (see below, Section 6 and [1] for details).

Theorem 4.13 *The $\mathbf{R}$-split level is sub-additive that is, for A and B two mixed Hodge structures in $\mathbf{R}$-MHS (resp. $\mathbf{C}$-MHS) and $H \in \operatorname{Ext}^1_{\mathbf{R}\text{-}MHS}(B, A)$ (resp. $\operatorname{Ext}^1_{\mathbf{C}\text{-}MHS}(B, A)$),*

$$\alpha(H) \geq \alpha(A) + \alpha(B).$$

P r o o f. The exact sequence given by the extension leads to the exact sequence (4.1) in the category of coherent sheaves. Thus, we have $\mathrm{c}_2(\xi_{\mathbf{P}^2}(H)) + \mathrm{c}_2(\mathcal{T}) = \mathrm{c}_2(\xi_{\mathbf{P}^2}(A)) + \mathrm{c}_2(\xi_{\mathbf{P}^2}(B))$. Since the support of $\mathcal{T}$ is at least 2-codimensional, $\mathrm{c}_2(\mathcal{T}) \leq 0$ (see [7], Chapter 2, Formula (2.8)), what allows us to conclude. □

In particular, it means that if $A \to H$ (resp. $H \to B$) is an injective (resp. surjective) morphism of mixed Hodge structures we have $\alpha(H) \geq \alpha(A)$ (resp. $\alpha(H) \geq \alpha(B)$).

Since every mixed Hodge structure can be written as a successive extension of pure Hodge structures whose $\mathbf{R}$-split level is 0, we have:

Corollary 4.14 *For each $H \in \mathbf{R}$-MHS, $\alpha(H) \geq 0$.*

Moreover, the $\mathbf{R}$-split level generalizes the notion of $\mathbf{R}$-split mixed Hodge structure:

Proposition 4.15 *A mixed Hodge structure is $\mathbf{R}$-split if and only if its $\mathbf{R}$-split level is 0.*

P r o o f. If H is a $\mathbf{R}$-split mixed Hodge structure, its associated Rees vector bundle is a direct sum of the Rees bundles associated with its direct summands $I^{p,q}$, which are endowed with (complex) mixed Hodge structures by the induced filtrations. Since the length of these mixed Hodge structures is 0, the associated bundles are trivial and so is their direct sum, hence $\alpha(H) = 0$.

The converse is a consequence of the fact that the moduli space of holomorphic bundles on $\mathbf{P}^2$, trivial on the line at infinity $\mathbf{P}^1_0$, with a fixed trivialization there, and with second Chern class zero is reduced to a point (see [6]). Let H be a mixed Hodge structure whose $\mathbf{R}$-split level is 0. By the proof of Proposition 3.4, $\xi_{\mathbf{P}^2}(H)$ can be endowed with a trivialization on $\mathbf{P}^1_0$, and hence, by the cited result, is trivial. Then, the fact that $\xi_{\mathbf{P}^2}(H)$ is $\mathbf{T}^\tau$-equivariant yields a decomposition into $I^{p,q}$ spaces with the required property. □

4.4 Mixed Hodge structures and instantons bundles

By a result of Donaldson [6], there is a natural one-to-one correspondence between the moduli space of rank r holomorphic bundles on $\mathbf{P}^2$, trivial on the line $\mathbf{P}^1_0$ and with a fixed framing here, whose second Chern class is n and the framed moduli space of instantons on the sphere $S^4 = \mathbf{R}^4 \cup \{\infty\}$ which parametrizes anti-self-dual connections on a principal $SU(r)$-bundle of charge n modulo gauge transformations γ with $\gamma_\infty = id$.

Remark now that once we have chosen an isomorphism $V_0 \xrightarrow{\sim} \bigoplus_r Gr^r_W V_0$, a mixed Hodge structure on V_0, (H, φ), where $\varphi : H_{\mathbf{C}} \xrightarrow{\sim} V_0$ is an isomorphism (see Section 3.4), canonically yields a Rees bundle $\xi_{\mathbf{P}^2}(H)$ which is framed on $\mathbf{P}^1_0$, namely a Rees bundle with an isomorphism $\xi_{\mathbf{P}^2}(H)|_{\mathbf{P}^1_0} \xrightarrow{\sim} \bigoplus_r Gr^r_W V_0 \otimes \mathcal{O}_{\mathbf{P}^1_0}$ which is uniquely determined by the starting choice. Thus, we have

Theorem 4.16 *To each isomorphism class of framed mixed Hodge structure H corresponds a unique isomorphism class of instanton on S^4 whose charge is the $\mathbf{R}$-split level of H, $\alpha(H)$.*

This description of mixed Hodge structures in terms of Yang-Mills instantons through the description of [6], which uses the ADHM construction, will be the subject of a forthcoming paper. Note that the Rees bundles associated with mixed Hodge structures are instantons in the sense of [13].

5 Examples of calculation of the R-split level

In this section, we first collect general considerations about the $\mathbf{R}$-split level and next we explicit it when associated to the mixed Hodge structures on the cohomology of some algebraic curves.

According to Deligne [3], [4], the cohomology groups of algebraic varieties, here separated schemes of finite type over $\mathbf{C}$, are endowed with natural mixed Hodge structures. For X such an algebraic variety, we will consider, for each integer l, the $\mathbf{R}$-split levels of the $l\mathrm{th}$ group of cohomology

$$\alpha_l(X) = \alpha((H^l(X, \mathbf{C}), W_\bullet, F^\bullet, \overline{F}^\bullet)).$$

For certain varieties these invariants are trivial. The mixed Hodge structures of length lower than 2 are indeed $\mathbf{R}$-split. It is the case for pure Hodge structures and hence for the Hodge structures on the cohomology groups of smooth projective algebraic varieties or of compact Kähler varieties. The lengths of the mixed Hodge structures on the cohomology of weighted projective spaces (see [5]) and of varieties with logarithmic singularities (see [22]) are lower than 2; these mixed Hodge structures are therefore $\mathbf{R}$-split.

The construction of mixed Hodge structures is functorial. Let X be an algebraic variety, let $\pi : X' \to X$ be a resolution of singularities, let $j : X \to \overline{X}$ be a compactification and let $S = \overline{X} \setminus X$. By [3], [4], we can find a compatible smooth compactification $\overline{j} : X' \to \overline{X}'$ and a morphism $\overline{\pi} : \overline{X}' \to \overline{X}$ making the square below cartesian:

$$\begin{tikzcd}
X' \arrow[r, "\overline{j}"] \arrow[d, "\pi"] & \overline{X}' \arrow[d, "\overline{\pi}"] & \\
X \arrow[r, "j"] & \overline{X} & S. \arrow[l, "i"] \arrow[ul, "\overline{i}"']
\end{tikzcd}$$

By [4, Proposition 8.2.6], π induces an epimorphism of mixed Hodge structures on the cohomology, and j induces a monomorphism. Thus, we have, for each integer l,

$$\alpha_l(X) \geq \alpha_l(X') \quad \text{and} \quad \alpha_l(X) \geq \alpha_l(\overline{X}).$$

Let us focus our attention on curves. The curve X' is now the normalization of X. The mixed Hodge structures on 0th and second cohomology groups are pure [4, Proposition 8.1.20]. Let us describe it on the first cohomology groups, which is given by the hypercohomology of the complex

$$\big[\mathcal{O}_{\overline{X}} \xrightarrow{d} \pi_* \Omega^1_{\overline{X}'}(\log S)\big].$$

The weight and Hodge filtrations on $H^1(X, \mathbf{C}) = \mathbf{H}^1\big(\big[\mathcal{O}_{\overline{X}} \xrightarrow{d} \pi_* \Omega^1_{\overline{X}'}(\log S)\big]\big)$ are respectively described by

Lemma 5.1 ([4, Lemme 10.3.11])

(1) $W^1(H^1(X, \mathbf{R})) = \mathrm{Im}(H^1(\overline{X}, \mathbf{R}) \to H^1(X, \mathbf{R}))$ *and* $W^0(H^1(X, \mathbf{R})) = \mathrm{Ker}(H^1(\overline{X}, \mathbf{R}) \to H^1(\overline{X}', \mathbf{R}))$.

(2) *The spectral sequence defined by the naive fitration of* $\big[\mathcal{O}_{\overline{X}} \xrightarrow{d} \pi_* \Omega^1_{\overline{X}'}(\log S)\big]$ *degenerates into the Hodge filtration of* $H^*(X, \mathbf{C})$.

In order to compute the $\mathbf{R}$-split level of the mixed Hodge structures on the cohomology groups, we shall compute the *period matrix*, which gives the coordinates of $F^1 H^1(X, \mathbf{C})$, using the preceding lemma, and then look at the intersection between the sub-vector space this matrix determines and its conjugate with respect to the underlying real structure.

5.1 Curves of genus 0

We consider a first non trivial example: a non-complete nodal curve of genus 0. Let (m_1, m_2, p_1, q_1) be four distinct points of $\mathbf{P}^1_{\mathbf{C}}$ and let X be the non-complete singular curve obtained by gluing p_1 and q_1 together and removing the m_i (for a justification of the identification of the points see [19], Chapter IV, Part 3). Here we have $S = m_1 \coprod m_2$. Let u be a coordinate on $\mathbf{P}^1_{\mathbf{C}} \backslash \{\infty\}$. One easily verifies that

Lemma 5.2 *$F^1 H^1(X, \mathbf{C})$ is generated by the 1-form* $\omega = \big(\frac{1}{u - m_1} - \frac{1}{u - m_2}\big) du$.

Since the mixed Hodge structure on the first cohomology group of such a curve is an extension of a Tate Hodge structure of weight 1 by a Tate Hodge structure of weight 0 (see Section 6.1), we have $h^{0,0} = h^{1,1} = 1$ and the remaining Hodge numbers are zero. Since $s^{1,0} + s^{1,1} = h^{1,0} + h^{1,1} = \dim_{\mathbf{C}} F^1 H^1(X, \mathbf{C})$, $s^{1,0} = s^{0,1}$ by symmetry and $\sum_{p,q} s^{p,q} = \mathrm{rk}_{\mathbf{C}} H^1(X, \mathbf{C})$, the $\mathbf{R}$-split level is completely determined by the integer $s^{1,1}$. To know $s^{1,1}$ we have to compute the dimension of the intersection of the subspace F^1 with its conjugate with respect to the real structure inherited from $H^1(X, \mathbf{R})$. The Hodge filtration induces a filtration on the dual of the first cohomology group using the isomorphism

$$H^1_{DR}(X, \mathbf{C}) \cong H^1_{Betti}(X, \mathbf{C}) \cong H_1(X, \mathbf{C})^* = (H_1(X, \mathbf{R}) \otimes \mathbf{C})^*.$$

Let us choose a basis γ_0, γ_1 of $H_1(X, \mathbf{R})$. Let γ_0 be a positively oriented loop whose homology class is nonzero in X but vanish in $X \cup m_1$ and γ_1 be the loop formed by a path from p_1 to q_1 in X' by identifying these points.

We can now compute the coordinates of the Hodge filtration with respect to this basis:

$$\langle w, \gamma_0 \rangle = \int_{\gamma_0} w = 2\pi i, \quad \text{and,} \quad \langle w, \gamma_1 \rangle = \int_{\gamma_1} w = \left[\log \left(\frac{u - m_1}{u - m_2} \right) \right]_{p_1}^{q_1} = \log(q_1, p_1, m_1, m_2),$$

where (q_1, p_1, m_1, m_2) is the cross-ratio of the four points. Since the action of $PGL(1)$ on $\mathbf{P}^1_{\mathbf{C}}$ is transitive on the triples of points, we can always suppose that $m_1 = 0, m_2 = 1, p_1 = \infty$. X is hence completely determined by $q_1 \in \mathbf{C} \setminus \{0, 1\}$ and will be denoted by X_{q_1}. Thus, the period matrix is $\big(2\pi i \quad \log\big(\frac{q_1}{q_1 - 1}\big)\big)$. The intersection of F^1 with its conjugate is nonzero if and only if $\frac{1}{2\pi i} \log\big(\frac{q_1}{q_1-1}\big)$ is real that is if and only if q_1 belongs to the real line $\mathrm{Re}_{\frac{1}{2}}$ of complex numbers whose real part is $\frac{1}{2}$. So, when $q_1 \in \mathrm{Re}_{\frac{1}{2}}$, we have $s^{1,1} = 1$. Otherwise $s^{1,1} = 0$. We thus obtain:

Proposition 5.3 *The $\mathbf{R}$-split level of the first cohomology group of the curve X_{q_1} is given by*

$$\begin{cases} \alpha_1(X_{q_1}) = 0 & \text{if} \quad q_1 \in \mathrm{Re}_{\frac{1}{2}}, \\ \alpha_1(X_{q_1}) = 1 & \text{otherwise.} \end{cases}$$

Note that one can easily generalize the preceding calculation with arbitrary many removed or identified points on $\mathbf{P}^1_0$.

Remark 5.4 When we consider a family of curves parametrized by $q_1 \in S$, where S is a variety, the $\mathbf{R}$-split level jumps in real dimensions. It reflects the behaviour of the intersection between the vector spaces given by the family of Hodge filtrations, which is holomorphic, and the family of its conjugate filtrations, which is anti-holomorphic.

5.2 Curves of genus 1

Let $(m_1, \dots, m_k, p_1, \dots, p_l, q_1, \dots q_l)$ be $k + 2l$ distinct points of a complete smooth curve of genus 1 and let X be the curve obtained by gluing p_i and q_i for each $i \in [1, l]$ and removing the m_i's. By Lemma 5.1, the Hodge numbers of $H^1(X, \mathbf{C})$ are $h^{0,0} = l$, $h^{1,0} = h^{0,1} = 1$ and $h^{1,1} = k - 1$. The original complete smooth curve is isomorphic to the quotient of $\mathbf{C}$ by the lattice $\mathbf{Z} + \tau\mathbf{Z}$, where $\tau \in \mathbf{Z}$ and $\mathrm{Im}(\tau) > 0$. Let u be a coordinate on $\mathbf{C}$. To explicit the mixed Hodge structure on the first cohomology group, we need the following result:

Proposition 5.5 ([12]) *The function $\Psi : z \mapsto \sum_{i=1}^{i=m-1} \lambda_i \frac{d}{du} \log(\theta(u - a_i)) + C$, where the $\{\lambda_i\}_{i \in [1, m-1]}$ and C are complex numbers such that $\sum_{i=1}^{i=m-1} \lambda_i = 1$, is $\Lambda_{\mathbf{Z}}$-periodic with simple poles at the points $a_i + \frac{1}{2}(1 + \tau)$ and residus λ_i, where θ is the theta function on the elliptic curve given by $\Lambda_{\mathbf{Z}}$, $\theta(u) = \sum_{n \in \mathbf{Z}} \exp(\pi\sqrt{-1}n^2 u + 2\pi\sqrt{-1}n\tau)$.*

Hence, we can choose following generators for $F^1H^1(X, \mathbf{C})$ (note that they all come from $F^1H^1(X', \mathbf{C})$): $\omega_0 = du$ and $\big\{\omega_i = d\log\big(\theta\big(u - m_i - \frac{1}{2}(1+\tau)\big) / \theta\big(u - m_i - \frac{1}{2}(1+\tau)\big)\big)\big\}_{i \in [1, k-1]}$. Let $\gamma_0, \gamma_1, \dots, \gamma_k, \eta_1, \dots \eta_l$ be generators of $H_1(X, \mathbf{R})$ such that γ_0 is the Poincaré dual of ω_0, such that for each $j \in [1, k]$ γ_i is null-homologue in $X \cup m_i$, and, for each $j \in [1, l]$, η_j represents a loop obtained by identifying p_j with q_j. For $i \in [1, k-1]$, the integration of ω_i along η_j gives

$$\langle \omega_i, \eta_j \rangle = \int_{\eta_j} \omega_i = \int_{p_j}^{q_j} \omega_i = \log \left(\frac{\theta\big(q_j - m_i - \frac{1}{2}(1+\tau)\big)}{\theta\big(q_j - m_{i+1} - \frac{1}{2}(1+\tau)\big)} \Big/ \frac{\theta\big(p_j - m_i - \frac{1}{2}(1+\tau)\big)}{\theta\big(p_j - m_{i+1} - \frac{1}{2}(1+\tau)\big)} \right),$$

number which we denote by $\log(\theta(m_i, m_{i+1}, p_j, q_j))$. The $k \times (k + l + 1)$-period matrix is thus given by:

$$M = \begin{pmatrix} 1 & 0 & \dots & \dots & \dots & 0 & | & 0 & \dots & 0 \\ \lambda_1 & 1 & -1 & 0 & \dots & 0 & | & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & | & \dots & \log(\theta(m_i, m_{i+1}, p_j, q_j)) & \dots \\ \lambda_{k-1} & 0 & \dots & 0 & 1 & -1 & | & \dots & \dots & \dots \end{pmatrix}.$$

As mentioned above, for curves, the number α_1 is completely determined by $s^{1,1}$.

Proposition 5.6 *The $\mathbf{R}$-split level of the first cohomology group of X is determined by the integer*

$$s^{1,1} = 2k - rk_{\mathbf{C}}\left(\frac{M}{\overline{M}}\right).$$

6 Extentions of Tate Hodge structures and higher extensions

Let us consider extensions in the abelian category of mixed Hodge structures. We will only consider *separated* extensions of mixed Hodge structures that is congruence classes of extensions of the type

$$0 \longrightarrow A \xrightarrow{i} H \xrightarrow{\pi} B \longrightarrow 0,$$

where the highest weight of A is less than the lowest weight of B. It is shown in [1] that the abelian group $\mathrm{Ext}^1_{\mathbf{Z}\text{-MHS}}(B, A)$ is naturally isomorphic to a generalized torus, the 0th Jacobian of the mixed Hodge structure $\mathrm{Hom}(B, A)$,

$$\mathrm{Ext}^1_{\mathbf{Z}\text{-MHS}}(B, A) \cong J^0\mathrm{Hom}(B, A) = \mathrm{Hom}(B, A)_{\mathbf{C}}/(F^0\mathrm{Hom}(B, A) + \mathrm{Hom}(B, A)_{\mathbf{Z}}), \tag{6.1}$$

where $\mathrm{Hom}(B, A)_{\mathbf{C}}$ (resp. $\mathrm{Hom}(B, A)_{\mathbf{Z}}$) is the complex vector space (resp. the underlying lattice) of the mixed Hodge structure $\mathrm{Hom}(B, A)$.

One directly translates the proof of (6.1) in [1] to prove that, in $\mathbf{R}$-MHS,

$$\mathrm{Ext}^1_{\mathbf{R}\text{-MHS}}(B, A) \cong \mathrm{Hom}(B, A)_{\mathbf{C}}/(F^0\mathrm{Hom}(B, A) + \mathrm{Hom}(B, A)_{\mathbf{R}}). \tag{6.2}$$

To understand extensions in $\mathbf{C}$-MHS, one remarks that an extension of B by A is determined by both decreasing filtrations $F^\bullet$ and $\hat{F}^\bullet$ on $A_{\mathbf{C}} \oplus B_{\mathbf{C}}$. Hence $(\mathrm{Hom}(B, A)_{\mathbf{C}}/(F^0\mathrm{Hom}(B, A)) \times (\mathrm{Hom}(B, A)_{\mathbf{C}}/(F^0\mathrm{Hom}(B, A))$ is a parameter space for extensions, the first factor (resp. second) corresponding to the choice $F^\bullet$ (resp. $\hat{F}^\bullet$). Now, the group $(\mathrm{Hom}(B, A)_{\mathbf{C}}/(F^0\mathrm{Hom}(B, A))$ acts transitively and effectively on the congruence classes. We thus have

$$\mathrm{Ext}^1_{\mathbf{C}\text{-MHS}}(B, A) \cong \mathrm{Hom}(B, A)_{\mathbf{C}}/F^0\mathrm{Hom}(B, A). \tag{6.3}$$

The equivalences of categories stated above yield the isomorphisms

$$\mathrm{Ext}^1_{L\text{-MHS}}(B, A) \cong \mathrm{Ext}^1_{\mathrm{Bun}_{\mathbf{P}^1_0\text{-sst},\,\mu=0}(\mathbf{P}^2_{\mathbf{C}}/\mathbf{T}^L)}(\xi_B, \xi_A),$$

where L is $\mathbf{C}$ or $\mathbf{R}$ and $\mathbf{T}^L$ refers to, respectively, $\mathbf{T}$ and $\mathbf{T}^\tau$. These isomorphisms give a description of the extensions of mixed Hodge structures in terms of extensions of equivariant coherent sheaves.

We remark that the description of the mixed Hodge structures in terms of Rees bundles allows us to use the $\mathbf{R}$-split level to define stratifications of the extensions groups

$$\mathrm{Ext}^1_{L\text{-MHS}}(B, A) = \bigsqcup_{\alpha(A)+\alpha(B) \le \alpha \le \alpha_{max}} \mathrm{Ext}^1_{L\text{-MHS}}(B, A)_\alpha.$$

Here $\mathrm{Ext}^1_{L-\mathrm{MHS}}(B, A)_\alpha$ denotes the extensions whose $\mathbf{R}$-split level is α. As we shall see in the example below, some strata could be empty. They are not subgroups in general. The upper bound α_{max} could be explicitly computed in combinatorial terms of the Hodge filtrations of A and B.

6.1 Extension of Tate's Hodge structures

Consider now extensions of two Tate's Hodge structures, namely elements of $\mathrm{Ext}^1_{L\text{-MHS}}(T\langle p\rangle, T\langle q\rangle)$, with $L = \mathbf{C}$ or $\mathbf{R}$ and $p > q$. Geometrically, such a group arises when dealing with the first cohomology group of a non-complete nodal curve of genus 0 (see Section 5.1). By the preceding discussion, these extension groups are, respectively, $\mathbf{C}$ and $\mathbf{C}/\mathbf{R} \cong \mathbf{R}$ (note that in the category of $\mathbf{Z}$-MHS we get $\mathbf{C}/\mathbf{Z} \cong \mathbf{C}^*$). In this section we shall recover these groups by considering the corresponding extension groups of equivariant sheaves on $\mathbf{P}^2$.

The $\mathbf{R}$-split level of the mixed Hodge structure associated with an extension class of two Tate's structures could be else $\alpha = 0$, for the class given by the split extension, or $\alpha = (p-q)^2$ for the other classes, depending on the fact that F^p and $\hat{F}^p$ (or $\overline{F}^p$) are collinear, then the only non-zero $s^{k,l}$ numbers are $s^{p,p} = s^{q,q} = 1$, or not, then $s^{p,q} = s^{q,p} = 1$ and the other $s^{k,l}$ are zero (the only non-zero Hodge numbers are $h^{p,p} = h^{q,q} = 1$). For non-split extensions, the exact sequence in the category of coherent sheaves associated with the exact sequence in $\mathrm{Bun}_{\mathbf{P}^1_0\text{-sst},\mu=0}(\mathbf{P}^2_{\mathbf{C}}/\mathbf{T})$ corresponding by the Rees functor to the extension of mixed Hodge structures is

$$0 \longrightarrow \xi_A \xrightarrow{\ i\ } \xi_H \xrightarrow{\ \pi\ } \xi_B \otimes \mathcal{I}_{P_{12}} \longrightarrow 0,$$

where ξ_A, ξ_B are trivial line bundles and $\mathcal{I}_{P_{12}}$ is the ideal sheaf corresponding to the zero-dimensional subscheme of length α, $[P_{12}]$.

Let us compute $\mathrm{Ext}^1(\xi_B \otimes \mathcal{I}_{P_{12}}, \xi_A)$ in the category of coherent sheaves following [7], Chapter 2. The exact sequence for Ext groups associated with the $\mathcal{E}xt^l(\xi_B \otimes \mathcal{I}_{P_{12}}, \xi_A)$ has E_2 terms

$$E_2^{k,l} = H^k(X, \mathcal{E}xt^l(\xi_B \otimes \mathcal{I}_{P_{12}}, \xi_A)) \Longrightarrow \mathrm{Ext}^{k+l}(\xi_B \otimes \mathcal{I}_{P_{12}}, \xi_A).$$

This leads to the exact sequence

$$0 \longrightarrow H^1\left(\xi_B{}^{-1} \otimes \xi_A\right) \longrightarrow \mathrm{Ext}^1(\xi_B \otimes \mathcal{I}_{P_{12}}, \xi_A) \longrightarrow H^0(\mathcal{E}xt^1(\xi_B \otimes \mathcal{I}_{P_{12}}, \xi_A)) \longrightarrow H^2\left(\xi_B{}^{-1} \otimes \xi_A\right).$$

Since for a surface $\mathcal{E}xt^1(\mathcal{I}_{P_{12}}, \mathcal{O}_{\mathbf{P}^2_{\mathbf{C}}}) = \mathcal{O}_{[P_{12}]}$ and here $H^1\left(\xi_B{}^{-1} \otimes \xi_A\right) = H^2\left(\xi_B{}^{-1} \otimes \xi_A\right) = 0$, we have $\mathrm{Ext}^1\left(\xi_B \otimes \mathcal{I}_{P_{12}}, \xi_A\right) = H^0\left(\mathbf{P}^2_{\mathbf{C}}, \mathcal{O}_{[P_{12}]}\right) = \mathbf{C}$. Equivariant extension groups are given by the spectral sequence with E_2 terms

$$E_2^{k,l} = H^k(\mathbf{T}, \mathrm{Ext}^l(\mathcal{F}, \mathcal{G})) \Longrightarrow \mathrm{Ext}^{k+l}_{\mathbf{T}}(\mathcal{F}, \mathcal{G}).$$

Since reductive groups do not have higher cohomology, we get

$$\mathrm{Ext}^n_{\mathbf{T}}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Ext}^n(\mathcal{F}, \mathcal{G})^{\mathbf{T}}.$$

The equivariant extensions correspond thus to the sections of $\mathcal{O}_{[P_{12}]}$ which are invariant by the action of the torus, that is all the sections of $\mathcal{O}_{[P_{12}]}$, since P_{12} is fixed by the action. According to [7, Theorem 8, p. 37], an extension corresponding to an element η is free if and only if the section η generates the sheaf $H^0\left(\mathbf{P}^2_{\mathbf{C}}, \mathcal{O}_{[P_{12}]}\right)$, namely, the natural map $\mathcal{O}_{\mathbf{P}^2_{\mathbf{C}}} \to \mathcal{O}_{[P_{12}]}$ is onto. Free $\mathbf{T}$-equivariant extensions are hence classified by $\mathbf{C}^*$ (the zero extension corresponding to the non free split extension). Thus, we recover the stratification

$$\mathrm{Ext}^1_{\mathbf{C}\text{-MHS}}(T\langle p\rangle, T\langle q\rangle) \cong \mathbf{C} = \{0\}_{\alpha=0} \sqcup \mathbf{C}^*_{\alpha=(p-q)^2}.$$

The extensions in $\mathbf{R}$-MHS correspond to the non-zero τ-invariant sections of $O_{[P_{12}]}$, namely the real sections. We recover in this way the stratified decomposition

$$\mathrm{Ext}^1_{\mathbf{R}\text{-MHS}}(T\langle p\rangle, T\langle q\rangle) \cong \mathbf{R} = \{0\}_{\alpha=0} \cup \mathbf{R}^*_{\alpha=(p-q)^2}.$$

6.2 Higher extensions

Let us now consider the higher extension groups in the abelian category of complex and real mixed Hodge structures (we refer to [2] for definitions and settings). We compute these groups using, for each integer $n > 1$, the isomorphism

$$\mathrm{Ext}^n_{L\text{-MHS}}(B, A) \cong \mathrm{Ext}^n_{\mathrm{Bun}_{\mathbf{P}^1_0\text{-sst},\mu=0}(\mathbf{P}^2_{\mathbf{C}}/\mathbf{T}^L)}(\xi_B, \xi_A),$$

where L is $\mathbf{C}$ or $\mathbf{R}$.

Proposition 6.1 *Let A, B be two elements of L-MHS, with $L = \mathbf{C}$ or $\mathbf{R}$. Then, for each $n > 1$,*

$$\mathrm{Ext}^n_{L\text{-MHS}}(B, A) = 0.$$

Proof. Let $U = \mathbf{P}^2_{\mathbf{C}} \backslash \{P_{12}\}$ and denote by $i : U \to \mathbf{P}^2_{\mathbf{C}}$ the inclusion morphism. We will show it induces a monomorphism $i^* : \mathrm{Ext}^n_{\mathrm{Bun}_{\mathbf{P}^1_0\text{-sst},\mu=0}(\mathbf{P}^2_{\mathbf{C}}/\mathbf{T}^\tau)}(\xi_B, \xi_A) \to \mathrm{Ext}^n_{Coh(U/\mathbf{T}^\tau)}(i^*\xi_B, i^*\xi_A)$. Let η, η' be two classes of n-extensions such that $i^*\eta = i^*\eta'$. We choose a representative for each class which we denote by $(H_\bullet, f)$ and $(H'_\bullet, g)$ for short, where $(H_\bullet, f)$ means that we have the following exact sequence in $\mathrm{Bun}_{\mathbf{P}^1_0\text{-sst},\mu=0}\left(\mathbf{P}^2_{\mathbf{C}}/\mathbf{T}^\tau\right)$:

$$0 \longrightarrow \xi_A \xrightarrow{f_0} \xi_{H_1} \xrightarrow{f_1} \cdots \longrightarrow \xi_{H_n} \xrightarrow{f_n} \xi_B \longrightarrow 0 .$$

The congruence between $i^*\eta$ and $i^*\eta'$ is given, for each $k \in [1, n]$, by a morphism $\alpha_k : i^*\xi_{H_k} \to i^*\xi_{H'_k}$ such that, for each $k \in [0, n]$, with $\alpha_0 = id_{\xi_A|_U}$ and $\alpha_{n+1} = id_{\xi_B|_U}$, $\alpha_{k+1} \circ i^* f_k = i^* g_k \circ \alpha_k$. All the morphisms α_k are morphisms between holomorphic vector bundles over $\mathbf{P}^2_{\mathbf{C}} \backslash \{P_{12}\}$ and P_{12} is 2-codimensional. Therefore, by Hartog's theorem, they can be extended in an unique way to morphisms $\tilde{\alpha}_k$ on the whole space. We shall show that this yield a congruence in the category of bundles in which cokernels have been modified. We therefore have to show the commutativity of each square. Suppose we have shown the squares are commutative up to level k (the commutativity 0th is trivially verified). We can suppose f_k and g_k to be surjective; $\xi_{H_{k+1}}$ $\left(\text{resp. } \xi_{H'_{k+1}}\right)$ is then the cokernel of f_{k-1} (resp. g_{k-1}) in $\mathrm{Bun}_{\mathbf{P}^1_0\text{-sst},\mu=0}\left(\mathbf{P}^2_{\mathbf{C}}/\mathbf{T}^\tau\right)$. The following diagrams are commutative

$$\begin{tikzcd}
\xi_{H_k} \arrow[r] \arrow[d, "\tilde{\alpha}_k"] & Coker(f_{k-1}) \arrow[r, "\nu"] \arrow[d, dotted, "\beta_k"] & \xi_{H_{k+1}} = Coker(f_{k-1})^{**} \arrow[d, dotted, "\beta_k^{**}"] \\
\xi_{H'_k} \arrow[r] & Coker(g_{k-1}) \arrow[r, "\nu'"] & \xi_{H'_{k+1}} = Coker(f_{k-1})^{**},
\end{tikzcd}$$

where β_k^{**} is the canonical morphism between the double duals associated with β_k. Since $\beta_k^{**}|_U = \alpha_{k+1} = \tilde{\alpha}_{k+1}|_U$, the morphisms β_k^{**} and $\tilde{\alpha}_{k+1}$ coincide on the whole space, which proves the commutativity of the square in $\mathrm{Bun}_{\mathbf{P}^1_0\text{-sst},\mu=0}\left(\mathbf{P}^2_{\mathbf{C}}/\mathbf{T}^\tau\right)$.

Let us now compute the extension groups in the category of coherent sheaves on U. Since $i^*\xi_B$ is locally free, we have $\mathrm{Ext}^n_{Coh(U)}(i^*\xi_B, i^*\xi_A) = H^n(U, i^*(\xi_B^* \otimes \xi_A))$. The long exact sequence of cohomology groups

$$\cdots \longrightarrow H^n_{P_{12}}\left(\mathbf{P}^2_{\mathbf{C}}, \xi_B^* \otimes \xi_A\right) \longrightarrow H^n\left(\mathbf{P}^2_{\mathbf{C}}, \xi_B^* \otimes \xi_A\right) \longrightarrow H^n\left(U, i^*(\xi_B^* \otimes \xi_A)\right) \longrightarrow \cdots$$

and the vanishing of the local cohomology groups yield, for n positive, the isomorphisms $H^n(U, i^*(\xi_B^* \otimes \xi_A)) \cong H^n\left(\mathbf{P}^2_{\mathbf{C}}, \xi_B^* \otimes \xi_A\right)$. Let $\xi_C \cong \xi_B^* \otimes \xi_A$ be the Rees bundle associated with the mixed Hodge structure $C = B^* \otimes A$. Let r be the lowest integer such that $W_r C \neq 0$. C can be written as an extension of $W_r C$ by $C/W_r C$, whose length is strictly lower than the length of C, and $W_r C$ is a pure Hodge structure. In the category of coherent sheaves we have two exact sequences

$$0 \longrightarrow \xi_{W_r C} \xrightarrow{i} \xi_C \xrightarrow{\pi} \mathcal{C}oker(i) \longrightarrow 0,$$

and

$$0 \longrightarrow \mathcal{C}oker(i) \xrightarrow{\nu} \xi_{C/W_r C} \longrightarrow \mathcal{T} \longrightarrow 0 ,$$

where the support of the sheaf $\mathcal{T}$ is included in P_{12}. Using the fact that $\xi_{W_r C}$ is a direct sum of line bundles of degree zero, and hence, for $n > 0$, $H^n\left(\mathbf{P}^2_{\mathbf{C}}, \xi_{W_r C}\right) = 0$, and the long exact sequence of cohomology groups associated with the exact sequence above we get, for each $n > 0$, $H^n\left(\mathbf{P}^2_{\mathbf{C}}, \xi_C\right) \cong H^n\left(\mathbf{P}^2_{\mathbf{C}}, \mathcal{C}oker(i)\right)$. Since $H^n\left(\mathbf{P}^2_{\mathbf{C}}, \mathcal{T}\right) = 0$ for each $n > 0$, the long exact sequence of cohomology groups associated with the second exact sequence

$$\cdots \longrightarrow H^{n-1}\left(\mathbf{P}^2_{\mathbf{C}}, \mathcal{T}\right) \longrightarrow H^n\left(\mathbf{P}^2_{\mathbf{C}}, \mathcal{C}oker(i)\right) \longrightarrow H^n\left(\mathbf{P}^2_{\mathbf{C}}, \xi_{B/W_r B}\right) \longrightarrow \cdots$$

gives, for each $n > 1$, $H^n\left(\mathbf{P}^2_{\mathbf{C}}, \xi_C\right) \cong H^n\left(\mathbf{P}^2_{\mathbf{C}}, \mathcal{C}oker(i)\right) \cong H^n\left(\mathbf{P}^2_{\mathbf{C}}, \xi_{C/W_r C}\right)$. We can iterate the decomposition until we have a pure Hodge structure. So, for each $n > 1$, $H^n\left(\mathbf{P}^2_{\mathbf{C}}, \xi_C\right) = 0$. We thus obtain, with $L = \mathbf{C}, \mathbf{R}$,

$$\begin{aligned}
\mathrm{Ext}^n_{Coh(U/\mathbf{T}^L)}(i^*\xi_B, i^*\xi_A) &\cong \mathrm{Ext}^n_{Coh(U)}(i^*\xi_B, i^*\xi_A)^{\mathbf{T}^L} \\
&\cong H^n(U, i^*(\xi_B^* \otimes \xi_A))^{\mathbf{T}^L} \\
&\cong H^n\left(\mathbf{P}^2_{\mathbf{C}}, \xi_B^* \otimes \xi_A\right)^{\mathbf{T}^L} \\
&\cong 0,
\end{aligned}$$

which completes the proof. □

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