3×3 DECOMPOSITION OF CIRCULAR STRUCTURING ELEMENTS

Maria Vanrell, Jordi Vitrià

Universitat Autònoma de Barcelona
Computer Vision Center
08193 Bellaterra, Barcelona
e-mail:maria@upisun1.uab.es

ABSTRACT

This paper presents some results to decompose circular structuring elements into 3×3 elements. Decomposition allows to improve the expended time in computing morphological operations. Generally, the shape of the structuring element determines the image transformation. Morphological operations with disks can be used as shape and size descriptors. The optimal discrete approximation of a disk can not be decomposed into 3×3 factors. Therefore, for a given radius, we give a hexadecogon that can be decomposed, and which optimally fits a disk. Afterwards, we present the decomposition of the disk in terms of the hexadecagon parameters. The decomposition prime factors can be given by different families of basic structuring elements.

Keywords: mathematical morphology, structuring elements, disks decomposition

1. INTRODUCTION

Morphological operators [1, 2] lead to image transformations which depend on the shape of the structuring elements. The decomposition of the structuring element is an important step to improve their computation. If a given structuring element B can be decomposed as $B = B_1 \oplus B_2 \oplus ... \oplus B_n$, then dilations and erosions can be computed by the following expressions

$$I \ominus B = ((...((I \ominus B_1) \ominus B_2)...) \ominus B_n)$$

$$I \oplus B = ((...((I \oplus B_1) \oplus B_2)...) \oplus B_n)$$

By using this property, parallel computers can ameliorate time costs when computing morphological operations [3].

Decomposition of structuring elements has been deeply studied in previous works [4, 5]. There are some works in which the decomposition is guided by the shape of the structuring elements, since structuring element shape plays an important role on how the image is transformed by a morphological operator. In this work, we give a useful decomposition of isotropic structuring elements, which can be used to define shape and size descriptors.

The objective of this work is to give a 3×3 decomposition of disks. Firstly, we propose the shape of a decomposable disk based on some given parameters. Secondly, we compute the possible solutions for the decomposition. Finally, we analize the results and give the best decomposition for some specific criteria.

2. DISK DECOMPOSITION

In order to obtain a disk decomposition, firstly, we have to describe the shape of a discrete disk approximation. This problem has been widely studied in Computer Graphics. The best disk is obtained using an incremental generator of circles, which is based on a midpoint criterion to minimize the error. We can take it as the discrete disk approximation, and attempt to demonstrate whether it can be decomposed.

Previous works on this subject have proved some propositions to assure decomposability depending on the element shape [6, 7]. These works have considered convex and concave structuring elements, respectively. Taking into account the convex and concave properties, we can state that the most isotropic convex structuring element is a regular poligon of eight sides. The only way to improve this shape is to appoximate the disk using a concave structuring element. In both works chain code representation of the element boundary is used.

Assuming that, to decompose a concave structuring element requires that its concave boundaries have to be contained in the decomposition prime factors. Then, the unique concave boundary that can be a disk boundary, and also be contained by a 3×3 structuring element is shown in figure 1(a). We will refer to it as a Q_r concave boundary, following the notation in [7].

Hence, the boundary of an isotropic structuring el-

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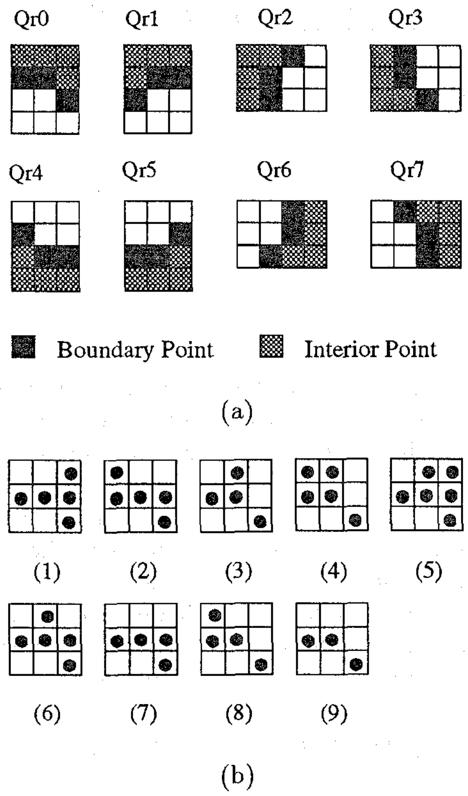

Figure 1: (a) Eight rotated versions of the concave boundary, which can form a decomposable disk boundary. (b) Basic 3×3 structuring elements containing the given boundaries

ement S, with boundaries contained in 3×3 prime factors, is given by the following chain code

$$S = Q_{r0}^{a} 0^{b} Q_{r1}^{a} 1^{c} Q_{r2}^{a} 2^{b} Q_{r3}^{a} 3^{c} Q_{r4}^{a} 4^{b} Q_{r5}^{a} 5^{c} Q_{r6}^{a} 6^{b} Q_{r7}^{a} 7^{c}$$

where the exponents indicate the number of repetition of each direction.

At this point, we can state that the optimal discrete approximation of a disk, we have mentioned before, can not always be expressed in this way. Thus, the optimal discrete disk appoximation can not be decomposed into 3×3 prime factors. The expression of S represents a discrete approximation of an hexadecagon, where a, b and c values depend on the radius of the disk we want to approximate.

We have constructed a constraint satisfaction algorithm to compute a, b and c values. It is based on a least-squares approach minimizing the following error expression:

$$E(a, b, c) = \sum_{x=1}^{R} (F(x) - \hat{F}(x))^2$$

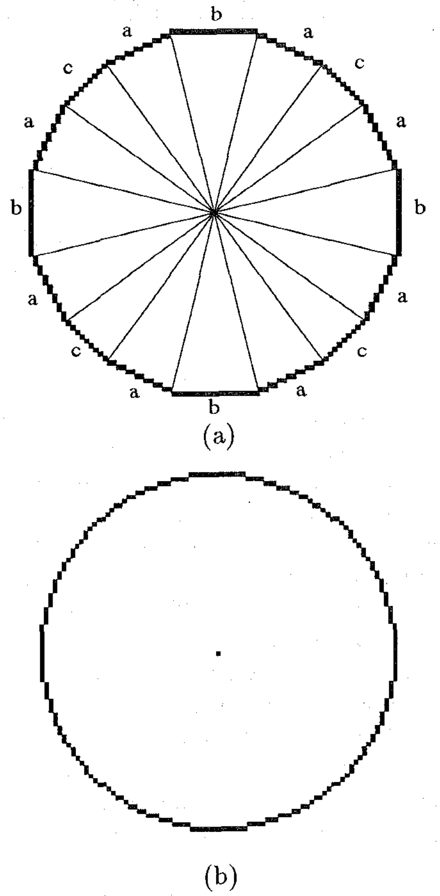

Figure 2: (a) Optimal hexadecagon approximation of a disk with radius=40 and $a = 7$, $b = 18$ and $c = 10$, for which there exists a 3×3 decomposition. (b) Optimal discrete approximation of a disk whith the same radius as (a), and for which there does not exist a 3×3 decomposition.

where $F(x) = \sqrt{R^2 - x^2}$, and $\hat{F}(x)$ is the value for the y coordinate on the polygon boundary that we want to construct. In figure 2 we show an example of the resulting hexadecagon for a given radius. In table 1 we give the values of a, b and c, for the first twenty-five discrete disk approximations.

Until now, we have defined the shape of a disk approximation which could have a 3×3 decomposition. Now, we want to prove its decomposability and find the expression of its decomposition.

Any disk S is decomposable if there exists a solution for $\mathbf{X}$ and $\mathbf{B}$, satisfying the following equations ([7])

Radius	1	2	3	4	5	6	7	8	9
a	0	0	0	0	0	0	1	1	2
b	0	2	2	2	4	4	4	4	4
c	1	1	2	3	3	4	2	3	1

Radius	10	11	12	13	14	15	16	17
a	1	1	2	2	2	2	3	3
b	6	6	6	6	8	8	8	8
c	4	5	3	4	4	5	3	4

Radius	18	19	20	21	22	23	24	25
a	3	3	3	4	4	3	4	4
b	8	10	10	10	10	12	12	12
c	5	5	6	4	5	8	6	7

Table 1: Values of a, b and c defining the optimal hexadecagon fitting a disk for a given radius.

$$
\begin{aligned}
\mathbf{\Theta X} &= \mathbf{Y} \\
\mathbf{Z} - \mathbf{B} &= \mathbf{\Omega X} \\
b_7 + b_0 + b_1 &= b_3 + b_4 + b_5 \\
b_1 + b_2 + b_3 &= b_5 + b_6 + b_7
\end{aligned}
$$

where $\mathbf{Y} = (a, a, a, a, a, a, a, a)$ and $\mathbf{Z} = (b, c, b, c, b, c, b, c)$, such that a, b and c depend on the radius of the disk. $\mathbf{\Theta}$ and $\mathbf{\Omega}$ are formed from the chain codes of the concave prime factors, in which the disk can be decomposed.

The solution will provide the decomposition of S in this form

$$S = x_1 A_1 \oplus ... \oplus x_n A_n \oplus B$$

where B represents a convex structuring element with chain code $(b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7)$, and whose decomposition has been solved in [6]. Also, it must be taken into account that n depends on the concave prime factors that we have selected for the decomposition.

All concave 3×3 prime factors containing boundaries in figure 1(a) are given in figure 1(b). In order to prove decomposability, we could consider all of them, and their rotations. But if we did it so, it would suppose to solve an undetermined system. To avoid this undetermination, we have considered each type of basic structuring elements separately. That is, we have solved nine linear systems and obtained one single solution for each one of them (see table 2 (a)).

3. RESULTS

In order to analize the solutions obtained in the previous section, we have considered two criteria. Firstly, it should be considered that every solution must accomplish $x_i \geq 0, b_i \geq 0 \ \forall i$. It implies that some constraints must be verified by the values of the disk parameters, they are shown in table 2 (b).

Secondly, the solution for every family of basic structuring elements can give different computation times for a basic morphological operation. We have assumed the number of translations to specify the cost of computing a basic operation. At first glance, families 7, 8 and 9 are the best due to their lowest cost. But at the same time they are the most restrictive and do not assure decomposability for an important number of disks.

Following these criteria we have selected families 1, 2 and 4 as the best solutions, since their constraints assure decomposability for the first 500 discrete disks.

Summarizing, we give the following steps to compute morphological operations with circular structuring elements, using a 3×3 decomposition:

1. We compute the optimal hexadecagon which approximates a disk S using the proposed algorithm. It gives the optimal values for a, b and c.

Family	$\mathbf{X} = (x_0, ..., x_{max})$	$\mathbf{B} = (k, m, k, m, k, m, k, m)$	
1,2,4	$x_i = a \ i = 0, ..., 4$	$k = b - 2a$	$m = c$
3	$x_i = a \ i = 0, ..., 4$	$k = b$	$m = c - a$
5	$x_i = a \ i = 0, ..., 7$	$k = b - 5a$	$m = c - 2a$
6	$x_i = a \ i = 0, ..., 7$	$k = b - 2a$	$m = c - 4a$
7	$x_i = a \ i = 0, ..., 7$	$k = b - 6a$	$m = c$
8	$x_i = a \ i = 0, ..., 7$	$k = b - 2a$	$m = c - 4a$
9	$x_i = a \ i = 0, ..., 7$	$k = b - 2a$	$m = c - 2a$

(a)

Family	Constraints		Translations
1,2,4	$b \geq 2a$	$c \geq 0$	$4(3a + b + c)$
3	$b \geq 0$	$c \geq a$	$4(3a + b + c)$
5	$b \geq 5a$	$c \geq 2a$	$4(5a + b + c)$
6	$b \geq 2a$	$c \geq 4a$	$4(4a + b + c)$
7	$b \geq 6a$	$c \geq 0$	$4(2a + b + c)$
8	$b \geq 2a$	$c \geq 4a$	$4(2a + b + c)$
9	$b \geq 2a$	$c \geq 2a$	$4(2a + b + c)$

(b)

Table 2: (a) Linear systems solutions for every family. (b) Constraints imposed by each family to obtain a valid decomposition solution. The number of translations needed to compute a morphological operation using the obtained decomposition.

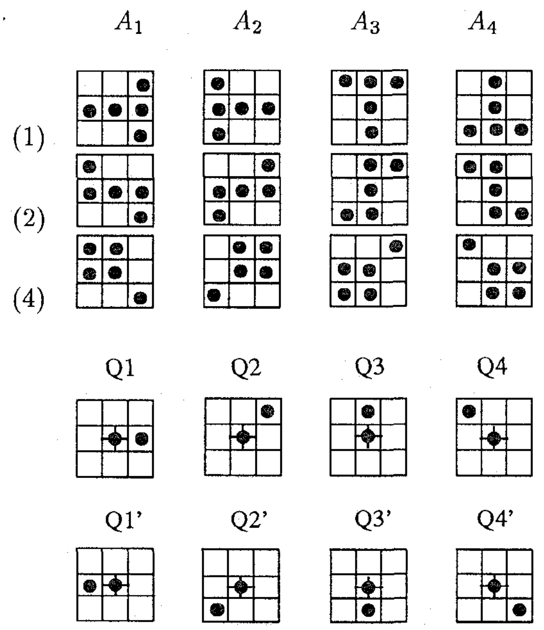

Figure 3: Basic structuring elements. Families 1, 2 and 4 give equivalent results decomposing concave boundaries. Elements Q_i are used to decompose convex boundaries.

2. We obtain the decomposition of S by the following expression

$$\begin{aligned} S &= aA_1 \oplus aA_2 \oplus aA_3 \oplus aA_4 \oplus \\ & (b-2a)Q1 \oplus cQ2 \oplus (b-2a)Q3 \oplus cQ4 \end{aligned}$$

The A_i's prime factors are the rotated versions of one element, which are grouped in families. The A_i family can be any one of 1, 2 or 4 (see figure 3). Whatever family we select, the result will be identical. The rest of prime factors, Q_i's, decompose the convex boundaries. To avoid the translations introduced by Q_1, Q_2, Q_3 and Q_4, we can use Q'_1, Q'_2, Q'_3 and Q'_4, and to do the following substitution

$$kQ_i = \frac{k}{2}Q_i \oplus \frac{k}{2}Q'_i$$

when k is odd, we will have to introduce an additional translation.

4. FUTURE WORK

In this article we make a simple approach to the disk decomposition problem, and we propose a practical solution.

It would be useful to improve the given results trying to find out, if it is possible, a general expression for the optimal a, b and c parameters, instead of the algorithmic solution given in this work.

On the other hand, and as a future work we could try to solve the proposed linear systems by combining more than one family of concave prime factors at the same time. Then we would have to analize the resulting solutions.

Finally, and from a more theoretical point of view it would be interesting to extend the work for bigger basic structuring elements. It seems clear that increasing the size of the basic decomposition elements, will allow to ameliorate the disk shape.

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